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Dispersionless and multicomponent BKP hierarchies with quantum torus symmetries

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ABSTRACT

In this article, we will construct the additional perturbative quantum torus symmetry of the dispersionless BKP hierarchy based on the W_∞ infinite dimensional Lie symmetry. These results show that the complete quantum torus symmetry is broken from the BKP hierarchy to its dispersionless hierarchies. Further a series of additional flows of the multicomponent BKP hierarchy will be defined and these flows constitute an N-folds direct product of the positive half of the quantum torus symmetries.

1. Introduction

The KP hierarchy is one of the most important integrable hierarchies [1] and it arises in many different fields of mathematics and physics such as the enumerative algebraic geometry, topological field and string theory. One of the most important studies on the KP hierarchy is the theoretical description of the solutions of the KP hierarchy using Lie groups and Lie algebras such as in [1–3], which is closely related to the infinite dimensional Grassmann manifolds [4,5].

In [6], Date, Jimbo, Kashiwara and Miwa extended their work on the KP hierarchy to the multicomponent KP hierarchy. In [7], Takasaki and Takebe derived a series of differential Fay identities for the multicomponent KP hierarchy from the bilinear identities and they showed that their dispersionless limits give rise to the universal Whitham hierarchy. Besides the multicomponent KP hierarchy, the extended and reduced multicomponent Toda hierarchies attract a lot of studies [8–10].

Additional symmetries have been studied in the explicit form of the additional flows of the KP hierarchy by Orlov and Shulman [11]. This kind of additional flows depend on dynamical variables explicitly and constitute a centerless $W_{1+\infty}$ algebra which is closely related to the theory of matrix models [12,13] by the Virasoro constraint and string equations. As a generalization of the Virasoro algebra, the Block algebra was studied a lot in the field of Lie algebras and it was studied intensively in Refs. [14–17]. In another paper [18], we give a novel Block type additional symmetry of the bigraded Toda hierarchy (BTH). Later we did a series of studies on integrable systems and Block algebras such as in [19–21]. After the quantization, the Block Lie algebra becomes the so-called quantum torus Lie algebra which can be seen in several recent references as [22,23].

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It is well known that the KP hierarchy has two sub-hierarchies, i.e. the BKP hierarchy and CKP hierarchy. About the BKP hierarchy, a lot of studies on additional symmetries have been done, such as additional symmetries of the BKP hierarchy [24], dispersionless BKP hierarchy [25,26], two-component BKP hierarchy [27] and its reduced hierarchies [20,21], supersymmetric BKP hierarchy [28] and so on. Dispersionless integrable systems [29] are very important in the study of all kinds of nonlinear sciences in physics, particularly in the application on the topological field theory [30] and matrix models [31,32]. In particular, dispersionless integrable systems have many typical properties such as the Lax pair, infinite conservation laws, symmetries and so on. On dispersionless integrable systems, we also did several studies such as [19,21]. With the above preparation, we should pay our attention to the quantum torus type additional symmetry of the multi-component BKP hierarchy [33] and dispersionless BKP hierarchy [25,34] from the points of the multi-component generalization of Lie algebras and the importance of dispersionless integrable systems.

In the next section, we firstly review the Lax equations of the BKP and dispersionless BKP hierarchies. In Section 3, under the basic Sato theory, we construct the additional symmetries of the dispersionless BKP hierarchy and the symmetries form a perturbative quantum torus Lie algebra. In Section 4, we recall the Lax equation of the multicomponent BKP hierarchy. In Section 5, we construct the additional symmetry of the multicomponent BKP hierarchy which turns out to be in an N-folds direct product of the infinite dimensional complete quantum torus Lie algebra.

2. BKP hierarchy and dispersionless BKP hierarchy

Similar to the general way in describing the classical the BKP hierarchy [1,2], we will give a brief introduction of the BKP hierarchy. We denote "$*$" as a formal adjoint operation defined by $A^* = \sum(-1)^i\partial^i a_i$ for an arbitrary scalar-valued pseudo-differential operator $A = \sum a_i\partial^i$, and $(AB)^* = B^*A^*$ for two scalar operators A, B. Based on the definition, the Lax operator of the BKP hierarchy is as

$$L_B = \partial + \sum_{i\geq 1} v_i\partial^{-i}, \tag{2.1}$$

such that

$$L_B^* = -\partial L_B\partial^{-1}. \tag{2.2}$$

Eq. (2.2) will be called the B type condition of the BKP hierarchy.

The BKP hierarchy is defined by the following Lax equations:

$$\frac{\partial L_B}{\partial t_k} = [(L_B^k)_+, L_B],\ \ k \in \mathbb{Z}_+^{\mathrm{odd}}. \tag{2.3}$$

The "+" in (2.3) means the nonnegative projection about the operator "∂" and "-" means the negative projection. Note that the $\partial/\partial t_1$ flow is equivalent to the $\partial/\partial x$ flow, therefore it is reasonable to assume $t_1 = x$ in the next sections. The Lax operator L_B can be generated by a dressing operator $\Phi_B = 1 + \sum_{k=1}^{\infty}\bar{\omega}_k\partial^{-k}$ in the following way

$$L_B = \Phi_B\partial\Phi_B^{-1}, \tag{2.4}$$

where Φ_B satisfies

$$\Phi_B^* = \partial\Phi_B^{-1}\partial^{-1}. \tag{2.5}$$

The dressing operator Φ_B needs to satisfy the following Sato equations

$$\frac{\partial\Phi_B}{\partial t_n} = -(L_B^n)_-\Phi_B, \quad n = 1, 3, 5, \ldots. \tag{2.6}$$

Introduce firstly the Lax function of dispersionless BKP hierarchy [25,34] as follows

$$L = k + u_1k^{-1} + u_3k^{-3} + \cdots + \cdots \tag{2.7}$$

where the coefficients $u_1, u_3, \ldots$ of the Lax function are the same as Eq. (4.1). The variables u_j are functions of the real variable x. The Lax function L can be written as

$$L = e^{ad\varphi}(k), \quad ad\varphi(\psi) = \{\varphi, \psi\} = \frac{\partial\varphi}{\partial k}\frac{\partial\psi}{\partial x} - \frac{\partial\varphi}{\partial k}\frac{\partial\psi}{\partial x}.$$

The dressing function has the following form

$$\varphi = \sum_{n=1}^{\infty}\varphi_{2n}k^{-2n+1}. \tag{2.8}$$

The dressing function φ is unique up to adding some Laurent series about the variable k with coefficients which do not depend on x. The dispersionless BKP hierarchy can be defined as follows.