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On a family of KP multi–line solitons associated to rational degenerations of real hyperelliptic curves and to the finite non–periodic Toda hierarchy

Simonetta Abenda

Dipartimento di Matematica, Università di Bologna, P.zza di Porta San Donato 5, I-40126 Bologna BO, Italy

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ABSTRACT

We continue the program started in Abenda and Grinevich (2015) of associating rational degenerations of M–curves to points in $Gr^{\mathrm{TNN}}(k,n)$ using KP theory for real finite gap solutions. More precisely, we focus on the inverse problem of characterizing the soliton data which produce Krichever divisors compatible with the KP reality condition when Γ is a certain rational degeneration of a hyperelliptic M–curve. Such choice is motivated by the fact that Γ is related to the curves associated to points in $Gr^{\mathrm{TP}}(1,n)$ and in $Gr^{\mathrm{TP}}(n-1,n)$ in Abenda and Grinevich (2015). We prove that the reality condition on the Krichever divisor on Γ singles out a special family of KP multi–line solitons (T–hyperelliptic solitons) in $Gr^{\mathrm{TP}}(k,n)$, $k\in[n-1]$, naturally connected to the finite non-periodic Toda hierarchy. We discuss the relations between the algebraic–geometric description of KP T–hyperelliptic solitons and of the open Toda system. Finally, we also explain the effect of the space–time transformation which conjugates soliton data in $Gr^{\mathrm{TP}}(k,n)$ to soliton data in $Gr^{\mathrm{TP}}(n-k,n)$ on the Krichever divisor for such KP solitons.

1. Introduction

Regular bounded KP $(n-k,k)$-line solitons are associated to soliton data $(\mathcal{K},[A])$ with $\mathcal{K}=\{\kappa_1<\cdots<\kappa_n\}$ and $[A]\in Gr^{\mathrm{TNN}}(k,n)$, the totally non–negative part of the real Grassmannian, which is a reduction of the infinite dimensional Sato Grassmannian [1]. The asymptotic properties of such solitons have been successfully related to the combinatorial structure of $Gr^{\mathrm{TNN}}(k,n)$ in [2–4].

On the other side, in principle, such soliton solutions may be obtained assigning Krichever data, which satisfy the KP reality condition, on rational degenerations of regular M–curves [5–8].

In [9], we have started the program of connecting such two areas of mathematics —- the theory of totally positive Grassmannians and the rational degenerations of regular M-curves - using the real finite–gap theory for regular bounded KP $(n-k,k)$-line solitons: we have associated to any soliton data $(\mathcal{K},[A])$ with $[A]\in Gr^{\mathrm{TP}}(k,n)$, a curve Γ which is the rational degeneration of a regular M-curve of genus $g=k(n-k)$ and a Krichever divisor $\mathcal{D}$ compatible with the reality conditions settled in [6].

In the present paper, we focus on the inverse problem: we choose Γ a given rational degeneration of an M–curve, we fix the point $P_+\in\Gamma$ where the KP wavefunction has its essential singularity and a local coordinate ζ on Γ such that

E-mail address: simonetta.abenda@unibo.it.

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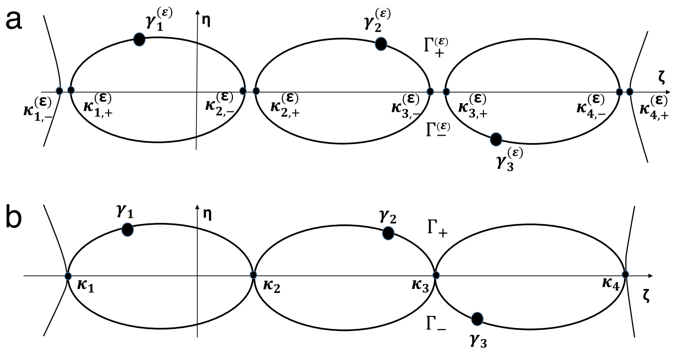

Fig. 1. Fig. 1(a): KP regular real quasi–periodic solutions on the genus 3 hyperelliptic curve $\Gamma^{(\epsilon)} = \Gamma_+^{(\epsilon)} \cup \Gamma_-^{(\epsilon)}$ are parametrized by 3-point divisors such that there is exactly one divisor point $\gamma_j^{(\epsilon)}, j \in [3]$ in each finite oval. Fig. 1(b): In the limit $\epsilon \to 0$, $\Gamma^{(\epsilon)}$ rationally degenerates to $\Gamma = \Gamma_+ \cup \Gamma_-$ and the limiting KP solution is a real bounded regular KP multi–soliton solution.

$\zeta^{-1}(P_+) = 0$, and we use the reality condition for the Krichever divisor to classify the regular bounded KP $(n-k,k)$–line solitons compatible with such algebraic–geometric setting, (Γ, P_+, ζ).

More precisely, we successfully investigate the case where Γ is obtained in the limit $\epsilon \to 0$ from regular real hyperelliptic curves with affine part $\Gamma^{(\epsilon)} = \{(\zeta,\eta) \ : \ \eta^2 = -\epsilon^2 + \prod_{j=1}^n (\zeta-\kappa_j)^2\}$, i.e.

$$\Gamma = \Gamma_+ \sqcup \Gamma_- = \left\{ (\zeta,\eta) \ : \ \eta^2 = \prod_{m=1}^{n} (\zeta - \kappa_m)^2 \right\}. \quad (1)$$

We choose Γ as in (1) because it is a desingularization of the curve associated in [9] to soliton data $(\mathcal{K}, [A])$, with $[A] \in Gr^{\rm TP}(n-1,n)$. Moreover, by a straightforward modification of the construction presented in [9], Γ is also associated to soliton data $(\mathcal{K}, [A])$, with $[A] \in Gr^{\rm TP}(1,n)$. We make the following ansatz:

(1) the number of phases n and the number k of divisor points belonging to the intersection of the finite ovals with Γ_+, the copy of $\mathbb{C}P^1$ containing the essential singularity of KP wavefunction, identifies the Sato finite dimensional reduction $Gr(k,n)$ corresponding to the KP solutions;
(2) the arithmetic genus of Γ, $n-1$, equals the dimension of the divisor and the dimension of the subvariety in $Gr(k,n)$ described by such KP soliton solutions;
(3) the real boundedness and regularity of the KP soliton solution due to the algebraic–geometric data implies that the soliton data are realizable in $Gr^{\rm TNN}(k,n)$.

The above ansatz is compatible both with real KP finite–gap theory, with Sato dressing of the vacuum with finite dimensional operators and the characterization of real regular bounded $(n-k,k)$–line solitons.

In the first part of the paper, we review some known facts about $(n-k,k)$–line soliton KP solutions and the finite–gap setting (Section 2), we justify the above ansatz by explicitly characterizing the KP–soliton data producing such divisor structure and call such KP–solitons T–hyperelliptic (Sections 3 and 4) and explain the relation with the results in [9] in the case $Gr^{\rm TP}(n-1,n)$ (Section 5). The main results of this part are:

(1) We identify the points in $Gr^{\rm TNN}(k,n)$ which correspond to real regular bounded KP $(n-k,k)$–line solitons with algebraic–geometric data on Γ;
(2) We prove that the divisor structure on Γ is compatible with the KP reality condition if and only if the soliton data $(\mathcal{K}, [A])$ correspond to points $[A] \in Gr^{\rm TP}(k,n)$, with

$$A^i_j = a_j k_j^{i-1}, \quad i \in [k],\ j \in [n], \qquad [a] \in Gr^{\rm TP}(1,n).$$

We call such KP solitons T–hyperelliptic and explicitly construct the KP–wavefunction on Γ. k divisor points belong to the intersection of the real ovals with Γ_+ and the remaining $(n-k-1)$ to the intersection of the real ovals with Γ_-. For instance the divisor structure shown in picture 1(b) corresponds to soliton data $\mathcal{K} = \{\kappa_1 < \cdots < \kappa_4\}$ and $[A] \in Gr^{\rm TP}(2,4)$ as above.

The special form of the KP τ–function associated to T–hyperelliptic solitons relates such class of solutions to the finite non–periodic Toda system [10]. Since Γ is related to the algebraic–geometric description of the finite non–periodic Toda hierarchy [11–13], it is then natural to investigate the relations between the algebraic–geometric approach for the two systems. The asymptotics of the KP wavefunction and of the Toda Baker–Akhiezer functions are different at the essential singularity P_+ since they are modeled respectively on regular finite gap KP theory and on the periodic Toda system.