Journal of Geometry and Physics 119 (2017) 187–192

Contents lists available at ScienceDirect

Journal of Geometry and Physics

journal homepage: www.elsevier.com/locate/geomphys

Circle actions on almost complex manifolds with isolated fixed points

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ARTICLE INFO

Article history:
Received 8 October 2015
Received in revised form 4 May 2017
Accepted 5 May 2017
Available online 11 May 2017

MSC:
37B05
47H10
58C30
58J20

Keywords:
Circle action
Almost complex manifold
Fixed point
Weight

ABSTRACT

In Jang (2014), the author proves that if the circle acts symplectically on a compact, connected symplectic manifold M with three fixed points, then M is equivariantly symplectomorphic to some standard action on $\mathbb{CP}^2$. In this paper, we extend the result to a circle action on an almost complex manifold; if the circle acts on a compact, connected almost complex manifold M with exactly three fixed points, then $\dim M = 4$. Moreover, the weights at the fixed points agree with those of a standard circle action on the complex projective plane $\mathbb{CP}^2$. Also, we deal with the cases of one fixed point and two fixed points.

1. Introduction

The purpose of this paper is to classify circle actions on compact almost complex manifolds with few fixed points. In [1], the author classifies a symplectic circle action on a compact symplectic manifold with three fixed points:

Theorem 1.1 ([1]). *Let the circle act symplectically on a compact, connected symplectic manifold M. If there are exactly three fixed points, then M is equivariantly symplectomorphic to some standard action on $\mathbb{CP}^2$ and the weights at the fixed points are $\{a+b, a\}$, $\{-a, b\}$, and $\{-b, -a-b\}$ for some positive integers a and b.*

In particular, the manifold has to be four-dimensional and the action must be Hamiltonian. In this paper, we extend the result to a circle action on an almost complex manifold.[1] The main result of this paper is the classification of a circle action on a compact almost complex manifold with at most three fixed points.

Theorem 1.2. *Let the circle act on a compact, connected almost complex manifold M.*

(1) *If there is exactly one fixed point, then M is a point.*
(2) *If there are exactly two fixed points, then either $\dim M = 2$ or $\dim M = 6$. If $\dim M = 2$, M is the 2-sphere and the weights at the fixed points are $\{a\}$ and $\{-a\}$ for some positive integer a. If $\dim M = 6$, then the weights at the fixed points are $\{-a-b, a, b\}$ and $\{-a, -b, a+b\}$ for some positive integers a and b.*

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[1] Throughout the paper, we assume the action preserves the almost complex structure.

http://dx.doi.org/10.1016/j.geomphys.2017.05.004
0393-0440/

Table 1
The classification of S^1-manifolds with few fixed points.

<table>
<tr><td colspan="3" rowspan="2">number of fixed points</td><td colspan="3">type of manifold</td></tr>
<tr><td>almost complex</td><td>symplectic</td><td>complex</td></tr>
<tr><td colspan="3">1</td><td colspan="3">$M = \{\mathrm{pt}\}$</td></tr>
<tr><td rowspan="4">2</td><td colspan="2">2 dim</td><td colspan="3">$M = S^2$</td></tr>
<tr><td rowspan="3">6 dim</td><td>invariants</td><td colspan="3">Chern (and Pontryagin) numbers and the Hirzebruch χ_y-genus are the same as S^6.</td></tr>
<tr><td>existence</td><td>S^6</td><td>not known</td><td>not known</td></tr>
<tr><td>uniqueness</td><td colspan="3">not known</td></tr>
<tr><td rowspan="3">3</td><td rowspan="3">4 dim</td><td>invariants</td><td colspan="3">Chern (and Pontryagin) numbers and the Hirzebruch χ_y-genus are the same as $\mathbb{CP}^2$.</td></tr>
<tr><td>existence</td><td colspan="3">$\mathbb{CP}^2$</td></tr>
<tr><td>uniqueness</td><td>not known</td><td>M is equivariantly symplectomorphic to $\mathbb{CP}^2$.</td><td>Possibly M is biholomorphic to $\mathbb{CP}^2$ by [3].</td></tr>
</table>

(3) *If there are exactly three fixed points, then* $\dim M = 4$. *Moreover, the weights at the fixed points are* $\{a+b, a\}$, $\{-a, b\}$, *and* $\{-b, -a-b\}$ *for some positive integers a and b.*

Theorem 1.2 will follow immediately from Theorem 2.8 and Theorem 2.10.

We compare circle actions with few fixed points on almost complex manifolds, symplectic manifolds, and complex manifolds by providing a table. For this, assume that if the circle acts on an almost complex (symplectic, and complex) manifold M, then the action preserves the almost complex (symplectic, and complex, respectively) structure.

Since any symplectic or complex manifold is almost complex, Theorem 1.2 implies the same results on the dimension and the weights at the fixed points for any symplectic or complex manifold as for an almost complex manifold. Moreover, since the weights at the fixed points determine the Chern (and Pontryagin) numbers and the Hirzebruch χ_y-genus, those invariants are the same for the three types of manifolds. On the other hand, on the existence whether such a manifold exists or not and on the uniqueness if we can determine such a manifold up to diffeomorphism (symplectomorphism or biholomorphism, respectively), the answers depend on the type of the manifold. In Table 1, any manifold is compact and connected, any circle action on the manifold preserves the given structure, and dim denotes the dimension of the manifold M. To the author's best knowledge, the classification is as in Table 1. The more there are fixed points, the harder the classification problem is; for the classification of an almost complex S^1-manifold with four fixed points in low dimensions, see [2]. Note that in Table 1, when M is complex and there are three fixed points, then $\dim M = 4$ by Theorem 1.2, and there is a possibility that M is biholomorphic to $\mathbb{CP}^2$ by the result of [3], in which Carrell, Howard, and Kosniowski classify holomorphic vector fields on complex surfaces with zeroes.

Now, we discuss the proof of Theorem 1.2. When there are one or two fixed points, then we give a complete proof in Section 2; see Theorem 2.8. For the case of three fixed points (Theorem 2.10), the idea of the proof is to adapt the proof of Theorem 1.1 in [1], since basically the same proof applies. To prove Theorem 1.1, in [1] the author uses the symplectic property in a number of places, and we carefully go through and eliminate this reliance. In particular, [1] adapts results for symplectic actions from other papers. We will have extensions of the results to almost complex S^1-manifolds in Section 2. If we extend to almost complex manifolds all the results that we need, then the same proof as in [1] goes through. Therefore, we conclude that a compact almost complex S^1-manifold M with three fixed points must have $\dim M = 4$ and the weights as described in Theorem 1.2. We clarify this at the end of this paper.

For a circle action on an almost complex manifold, there is an interesting and important conjecture by Kosniowski on the relationship between the dimension of a manifold and the number of fixed points [4].

Conjecture 1.3 ([4]). *Let the circle act on a* $2n$*-dimensional compact, connected almost complex manifold with k fixed points. Then* $n \leq f(k)$, *where* $f(k)$ *is a linear function in k.*

Kosniowski conjectures further that $f(k) = 2k$. Theorem 1.2 confirms that the conjecture is true if the number of fixed points is at most three. In general, Kosniowski's conjecture is challenging. Nevertheless, we verify the conjecture in a special case. For this, let the circle act on a compact almost complex manifold M with isolated fixed points. The *Chern class map* of M is the map

$$c_1(M) : M^{S^1} \longrightarrow \mathbb{Z},\ p \mapsto c_1(M)(p) \in \mathbb{Z},$$