



Heat trace for Laplace type operators with non-scalar symbols

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ABSTRACT

For an elliptic selfadjoint operator $P = -[u^{\mu\nu}\partial_\mu\partial_\nu + v^\nu\partial_\nu + w]$ acting on a fiber bundle over a compact Riemannian manifold, where $u^{\mu\nu}$, v^μ , w are $N \times N$ -matrices, we develop a method to compute the heat-trace coefficients a_r which allows to get them by a pure computational machinery. It is exemplified in any even dimension by the value of a_1 written both in terms of $u^{\mu\nu} = g^{\mu\nu}u$, v^μ , w or diffeomorphic and gauge invariants. We also address the question: when is it possible to get explicit formulae for a_r ?

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1. Introduction

We consider a compact Riemannian manifold (M, g) without boundary and of dimension d together with the nonminimal differential operator

$$P := -[u^{\mu\nu}(x)\partial_\mu\partial_\nu + v^\nu(x)\partial_\nu + w(x)] \quad (1.1)$$

which is a differential operator on a smooth vector bundle V over M of fiber \mathbb{C}^N where $u^{\mu\nu}$, v^ν , w are $N \times N$ -matrices valued functions. This bundle is endowed with a hermitian metric. We work in a local trivialization of V over an open subset of M which is also a chart on M with coordinates (x^μ) . In this trivialization, the adjoint for the hermitian metric corresponds to the adjoint of matrices and the trace on endomorphisms on V becomes the usual trace tr on matrices. Since we want P to be a selfadjoint and elliptic operator on $L^2(M, V)$, we first assume that $u^{\mu\nu}(x)\xi_\mu\xi_\nu$ is a positive definite matrix in M_N :

$$u^{\mu\nu}(x)\xi_\mu\xi_\nu \text{ has only strictly positive eigenvalues for any } \xi \neq 0. \quad (1.2)$$

We may assume without loss of generality that $u^{\mu\nu} = u^{\nu\mu}$. In particular $u^{\mu\mu}$ is a positive matrix for each μ and each $u^{\mu\nu}$ is selfadjoint.

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The asymptotics of the heat-trace

$$\text{Tr } e^{-tP} \underset{t \downarrow 0^+}{\sim} \sum_{r=0}^{\infty} a_r(P) t^{r-d/2} \tag{1.3}$$

exists by standard techniques (see [1, Section 1.8.1]), so we want to compute these coefficients $a_r(P)$.

While the spectrum of P is a priori inaccessible, the computation of few coefficients of this asymptotics is eventually possible. In physics, the operators P appeared in gauge field theories, string theory or the so-called non-commutative gravity theory (see for instance the references quoted in [2–4]). The knowledge of the coefficients a_r are important in physics. For instance, the one-loop renormalization in dimension four requires a_1 and a_2 .

When the principal symbol of P is scalar ($u^{\mu\nu} = g^{\mu\nu} \mathbb{1}_N$), there are essentially two main roads towards the calculation of heat coefficients (with numerous variants): the first is analytical and based on elliptic pseudodifferential operators while the second is more concerned by the geometry of the Riemannian manifold M itself with the search for invariants or conformal covariance. Compared with the flourishing literature existing when the principal symbol is scalar, there are only few works when it is not. One can quote for instance the case of operators acting on differential forms [5–8]. The first general results are in [9] or in the context of spin geometry using the Dirac operators or Stein–Weiss operators [10] also motivated by physics [2]. See also the approach in [11–19].

The present work has a natural algebraic flavor inherited from the framework of operators on Hilbert space comprising its own standard analytical part, so is related with the first road. In particular, it gives all ingredients to produce mechanically the heat coefficients. It is also inspired by the geometry *à la* Connes where $P = \mathcal{D}^2$ for a commutative spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$, thus has a deep motivation for noncommutative geometry.

Let us now enter into few technical difficulties.

While the formula for $a_0(P)$ is easily obtained, the computation of $a_1(P)$ is much more involved. To locate some difficulties, we first recall the parametrix approach, namely the use of momentum space coordinates $(x, \xi) \in T_x^*M$:

$$\begin{aligned} d_2(x, \xi) &= u^{\mu\nu}(x) \xi_\mu \xi_\nu, \\ d_1(x, \xi) &= -iv^\mu(x) \xi_\mu, \\ d_0(x) &= -w(x). \end{aligned}$$

Then we can try to use the generic formula (see [5])

$$a_r(P) = \frac{1}{(2\pi)^d} \frac{1}{-i2\pi} \int dx d\lambda d\xi e^{-\lambda} \text{tr} [b_{2r}(x, \xi, \lambda)] \tag{1.4}$$

where λ belongs to a anticlockwise curve \mathcal{C} around \mathbb{R}^+ and $(x, \xi) \in T^*(M)$. Here the functions b_{2r} are defined recursively by

$$\begin{aligned} b_0(x, \xi, \lambda) &:= (d_2(x, \xi) - \lambda)^{-1}, \\ b_r(x, \xi, \lambda) &:= - \sum_{\substack{r=j+|\alpha|+2-k \\ j < r}} \frac{(-j)^{|\alpha|}}{\alpha!} (\partial_\xi^\alpha b_j)(\partial_x^\alpha d_k) b_0. \end{aligned}$$

The functions b_{2r} , even for $r = 1$, generate typically terms of the form

$$\text{tr}[A_1(\lambda)B_1A_2(\lambda)B_2A_3(\lambda) \dots]$$

where all matrices $A_i(\lambda) = (d_2(x, \xi) - \lambda)^{-n_i}$ commute but do not commute a priori with B_i , so that the integral in λ is quite difficult to evaluate in an efficient way. Of course, one can use the spectral decomposition $d_2 = \sum_i \lambda_i \pi_i$ (depending on x and ξ) to get,

$$\sum_{i_1, i_2, i_3, \dots} \left[\int_{\lambda \in \mathcal{C}} d\lambda e^{-\lambda} (\lambda_{i_1} - \lambda)^{-n_{i_1}} (\lambda_{i_2} - \lambda)^{-n_{i_2}} (\lambda_{i_3} - \lambda)^{-n_{i_3}} \dots \right] \text{tr}(\pi_{i_1} B_1 \pi_{i_2} B_2 \pi_{i_3} \dots). \tag{1.5}$$

While the λ -integral is easy via residue calculus, the difficulty is then to recombine the sum.

This approach is conceptually based on an approximation of the resolvent $(P - \lambda)^{-1}$.

Because of previous difficulties, we are going to follow another strategy, using a purely functional approach for the kernel of e^{-tP} which is based on the Volterra series (see [20, p. 78],[4, Section 1.17.2]). This approach is not new and has been used for the same purpose in [9,2,3].

However our strategy is more algebraic and more in the spirit of rearrangement lemmas worked out in [21,22]. In particular we do not go through the spectral decomposition of $u^{\mu\nu}$ crucially used in [9] (although in a slightly more general case than the one of Section 4). To explain this strategy, we need first to fix a few notation points.

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