Journal of Geometry and Physics 116 (2017) 124–139

Contents lists available at ScienceDirect

Journal of Geometry and Physics

journal homepage: www.elsevier.com/locate/jgp

Recursion operators and bi-Hamiltonian structure of the general heavenly equation

M.B. Sheftel [a,*], D. Yazıcı [b], A.A. Malykh [c]

[a] Department of Physics, Boğaziçi University, Bebek, 34342 Istanbul, Turkey
[b] Department of Physics, Yıldız Technical University, Esenler, 34220 Istanbul, Turkey
[c] Department of Numerical Modelling, Russian State Hydrometeorological University, Malookhtinsky pr. 98, 195196 St. Petersburg, Russia

ARTICLE INFO

Article history:
Received 20 October 2015
Received in revised form 16 January 2017
Accepted 26 January 2017
Available online 6 February 2017

MSC:
35Q75
83C15
37K05
37K10

Keywords:
General heavenly equation
Lax pair
Recursion operator
Symplectic form
Hamiltonian operator
Jacobi identities

ABSTRACT

We discover two additional Lax pairs and three nonlocal recursion operators for symmetries of the general heavenly equation introduced by Doubrov and Ferapontov. Converting the equation to a two-component form, we obtain Lagrangian and Hamiltonian structures of the two-component general heavenly system. We study all point symmetries of the two-component system and, using the inverse Noether theorem in the Hamiltonian form, obtain all the integrals of motion corresponding to each variational (Noether) symmetry. We discover that in the two-component form we have only a single nonlocal recursion operator. Composing the recursion operator with the first Hamiltonian operator we obtain second Hamiltonian operator. We check the Jacobi identities for the second Hamiltonian operator and compatibility of the two Hamiltonian structures using P. Olver's theory of functional multi-vectors. Our well-founded conjecture is that P. Olver's method works fine for nonlocal operators. We show that the general heavenly equation in the two-component form is a bi-Hamiltonian system integrable in the sense of Magri. We demonstrate how to obtain nonlocal Hamiltonian flows generated by local Hamiltonians by using formal adjoint recursion operator.

1. Introduction

In paper [1], Doubrov and Ferapontov introduced the general heavenly equation (GHE)

$$\alpha u_{12}u_{34} + \beta u_{13}u_{24} + \gamma u_{14}u_{23} = 0, \qquad \alpha + \beta + \gamma = 0 \tag{1.1}$$

where α, β and γ are arbitrary constants satisfying one linear relation given above. Here $u = u(z_1, z_2, z_3, z_4)$ is a holomorphic function of four complex variables. They also obtained the Lax pair for Eq. (1.1)

$$\begin{aligned} X_1 &= u_{34}\partial_1 - u_{13}\partial_4 + \gamma\lambda(u_{34}\partial_1 - u_{14}\partial_3), \\ X_2 &= u_{23}\partial_4 - u_{34}\partial_2 + \beta\lambda(u_{34}\partial_2 - u_{24}\partial_3), \end{aligned} \tag{1.2}$$

where ∂_i means $\partial/\partial z_i$, while $u_{ij} = \partial^2 u/\partial z_i\partial z_j$.

There are only few examples of multi-dimensional integrable systems. The so-called heavenly equations make up an important class of such integrable systems since they are obtained by a reduction of the Einstein equations with Euclidean

* Corresponding author.
E-mail addresses: mikhail.sheftel@boun.edu.tr (M.B. Sheftel), yazici@yildiz.edu.tr (D. Yazıcı), andrei-malykh@mail.ru (A.A. Malykh).

http://dx.doi.org/10.1016/j.geomphys.2017.01.026
0393-0440/

(and neutral) signature for (anti-)self-dual gravity which includes the theory of gravitational instantons. All of these equations are of Monge–Ampère type, so that the only nonlinear terms are Hessian 2×2 determinants. General heavenly equation is an important example of such equations. An explicit description of ASD Ricci-flat vacuum metric governed by GHE, null tetrad and basis 1-forms for this metric were obtained in our paper [2]. Recently, Bogdanov [3] showed important relations between GHE and heavenly equations of Plebański and developed $\bar{\partial}$-dressing scheme for GHE in the context of the inverse scattering method. This stresses the importance of further study of the Doubrov–Ferapontov's general heavenly equation.

In this paper, we obtain recursion operators for symmetries of GHE (1.1) and discover its bi-Hamiltonian structure in a two-component form, where the single second-order PDE (1.1) is presented as an evolutionary system of two PDEs with two unknowns. Therefore, by the theorem of Magri [4] this general heavenly (GH) system is completely integrable.

While completing this paper, we became aware of the preprint [5] by A. Sergyeyev where he discovers a recursion operator for the one-component version of GHE, which coincides with the first one of our recursion operators, as an example of application of his general method for constructing recursion operators for dispersionless integrable systems.

If φ is a symmetry characteristic for (1.1), it satisfies the symmetry condition, "linearization" of Eq. (1.1)

$$\hat{A}\varphi = \alpha(u_{34}\varphi_{12} + u_{12}\varphi_{34}) + \beta(u_{24}\varphi_{13} + u_{13}\varphi_{24}) + \gamma(u_{23}\varphi_{14} + u_{14}\varphi_{23}) = 0. \tag{1.3}$$

A recursion operator maps any symmetry again into a symmetry and, as a consequence, it commutes with the operator $\hat{A}$ on solutions. For all other heavenly equations in the classification [1] of Doubrov and Ferapontov the symmetry condition has a two-dimensional divergence form which allows us to introduce partner symmetries [6–12], a powerful tool for finding recursion operators and noninvariant solutions [13,14] which are necessary for the construction of the famous gravitational instanton $K3$. It is easy to check that the symmetry condition (1.3) for the general heavenly equation cannot be presented in a two-dimensional divergence form but it can be presented in a three-dimensional divergence form.

Therefore, the method of partner symmetries does not work any more for GHE, so that we have to use here a different approach in order to find a recursion operator which could be regarded as a generalization of the method of partner symmetries. This approach is based on splitting each of the two Lax operators with respect to the spectral parameter λ in two operators and multiplying the first operator by the inverse of the second operator in each pair, obtained by the splitting in λ. This method was presented earlier in the first version of [5] using somewhat more geometric language. The idea of obtaining recursion operators from Lax pairs was used in 2003 in our paper on partner symmetries of the complex Monge–Ampère equation [7].

For a single-component equation (1.1), we discover three Lax pairs, related by discrete symmetries of both GHE and its symmetry condition, and three recursion operators corresponding to them. However, a two component form of Eq. (1.1) is not invariant under these permutations of indices, so two other recursion operators are related to two other 2-component systems. We do not consider them here because they are obtained from our two-component system just by the permutations of indices.

Another important property of heavenly equations is that they can be presented in a two-component evolutionary form as bi-Hamiltonian systems [15–17,11,12]. We show here that GHE also possesses this property. In a two-component form we construct a Lagrangian for this system and discover its symplectic and Hamiltonian structure. We obtain all point symmetries of this system and, using its Hamiltonian structure, apply the inverse Noether theorem to derive Hamiltonians generating all symmetry flows. Hamiltonians of the symmetry flows which commute with the GH flow are integrals of the motion for the GH system. Each such Hamiltonian is also conserved by all the symmetry flows that commute with the symmetry generated by this Hamiltonian. This procedure works only for variational (Noether) symmetries. Composing the recursion operator in a 2×2 matrix form with the Hamiltonian operator J_0 we generate a candidate for the second Hamiltonian operator $J_1 = RJ_0$. The property of J_1 to be Hamiltonian operator is equivalent to the recursion operator being hereditary (Nijenhuis) [18,19]. Since this property of our R is not known, we check directly the Jacobi identities for J_1, which is obviously skew-symmetric, and the compatibility of the two Hamiltonian operators J_0 and J_1. We find the corresponding Hamiltonian density H_0 such that the original general heavenly flow is generated by the action of J_1 on variational derivatives of the Hamiltonian functional $\mathcal{H}_0$, so that the GH system turns out to be a bi-Hamiltonian system. We demonstrate how applying the formal adjoint recursion operator $R^\dagger$ we can generate higher flows which are nonlocal symmetries of the system. We show how further local Hamiltonians can be constructed which generate nonlocal Hamiltonian flows.

In Section 2, we obtain two more Lax pairs for GHE in addition to the Doubrov–Ferapontov Lax pair. We show how to use these three Lax pairs for constructing three nonlocal recursion operators for GHE.

In Section 3, we present the general heavenly equation in a two-component evolutionary form and obtain a Lagrangian for this GH system.

In Section 4, we discover symplectic and Hamiltonian structure of the GH system.

In Section 5, we obtain all point Lie symmetries of the GH system and show how the corresponding Hamiltonians of the variational symmetry flows can be obtained from the inverse Noether theorem in a Hamiltonian form. These Hamiltonians are integrals of the motion along the original GH flow for all the variational symmetry flows that commute with the GH flow. Each Hamiltonian is also conserved along all the symmetry flows that commute with the flow generated by this Hamiltonian.

In Section 6, we derive a nonlocal recursion operator for the two component GH system.

In Section 7, composing the recursion operator with the first Hamiltonian operator J_0 we obtain the second nonlocal Hamiltonian operator J_1. We also find the corresponding Hamiltonian density which generates the GH system with the aid