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Applying the index theorem to non-smooth operators

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ABSTRACT

We give a simple way to extend index-theoretical statements from partial differential operators with smooth coefficients to operators with coefficients of finite Sobolev order.

In the theory of partial differential equations and in particular in index theory it is most common to consider only partial differential operators P with smooth coefficients, or, more generally, smooth pseudo-differential operators, which still map smooth sections to smooth sections. If P is moreover elliptic, this allows defining a parametrix of P, that is, an inverse up to infinitely smoothing operators. However, sometimes the assumption of smooth coefficients is not justified, e.g. in the theory of geodesic curves on some natural spaces of mappings. Here, to ensure some minimal amount of computability, one usually restricts the considerations to spaces of mappings of finite Sobolev or Hölder regularity instead of smooth mappings. Consequently, the induced metrics are also of finite regularity only, and so are the differential operators defined in terms of them, which appear in the geodesic equations. Often, as in [1], the proof of local existence and uniqueness for the geodesic equations requires an application of the index theorem to such an operator with Sobolev coefficients seen as a Fredholm map between Sobolev spaces $H^{k+p}(\pi) \to H^k(\pi)$ of sections of a vector bundle $\pi : E \to M$ over a compact manifold M. One could hope that the notion of abstract elliptic operators due to Atiyah [2] and elaborated on topological manifolds by Teleman [3] embark the case of partial differential operators with Sobolev coefficients. Amazingly enough, this does not seem to be the case, as one of the assumptions of the theorems in those articles is that multiplication with every $f \in C^0(M)$ is a bounded operator on the Hilbert spaces used, which is not true for any Sobolev Hilbert space except $L^2(M)$. Here we want to present a solution to the above problem, assuming coefficients of regularity increasing linearly with the order, a condition satisfied by the Laplacian. After the prepublication of the first version of the article, the author learned about the techniques presented in the third chapter of [4]. It would be interesting to compare the results of this note to the results about special first and second order pseudodifferential elliptic operators obtained there. The author wants to thank Ulrich Bunke for an interesting discussion on the topic of this note and Martin Bauer, Martins Bruveris, Peter Michor and an anonymous referee for many useful suggestions.

Conventions: We impose the convention $0 \in \mathbb{N}$, and for $T \in \mathbb{N}$ we define $\mathbb{N}_T := \mathbb{N} \cap [1, T]$. Throughout the article, M denotes an arbitrary smooth compact n-dimensional manifold. Contrary e.g. to [5], no smoothness assumption is made on

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the coefficients of a differential operator (as functions on the corresponding jet bundles) — they are a priori only assumed to be measurable.

Let M be a smooth compact manifold of dimension n, let $k, N \in \mathbb{N}$, and let π be an H^k vector bundle of rank N over M, i.e. there is a bundle atlas (U_t, D_t), $t \in \mathbb{N}_T$, with H^k trivialization changes. If g is an H^k vector bundle scalar product on π, ω is a volume form on M and ∇ is a metric connection on π with Christoffel symbols of regularity H^{k-1} in the bundle charts, then (π, g, ∇) is called a **metric H^k vector bundle with connection**. In this case we define in the usual way the norm $W^{l,q}(g, \nabla)$ (and the scalar product $H^l(g, \nabla)$ associated to $W^{l,2}(g, \nabla)$) on the Sobolev space $W^{l,q}(\pi)$ of $W^{l,q}$ sections of π for any $l \leq k$. A differential operator P of order $s \leq k$ on π is called k-**safe** iff for every U_t and every multiindex $i = (i_1, \ldots, i_r)$ there are $(N \times N)$-matrices $P_{i,t}$ with

$$(\mathrm{jet}^s D_t) \circ P \circ (D_t)^{-1} = \sum_{r=0}^{s} \sum_{1 \leq \ldots i_1 \leq i_2 \leq \ldots \leq i_r \leq n} P_{i,t} \partial_{i_1} \ldots \partial_{i_r},$$

and such that every matrix $P_{i,t}$ is of regularity $H^{p(i,k)}$ where

$$p(i, k) := \max \left\{ k - s, |i| - s + \left\lfloor \frac{n}{2} \right\rfloor + 1 \right\},$$

where, for $i \in \mathbb{N}$, $\lfloor i \rfloor$ is the Gauss bracket of i. The first thing to note is that $p(i, k) > \frac{n}{2}$ for all i with $|i| = s$, i.e., the leading coefficients are continuous, which means that ellipticity can be defined as usual. Obviously, the k-safe operators of degree s form a linear subspace $S_k^s(\pi)$ of the space of operators of degree s, which we equip with the scalar product $(\cdot, \cdot)$ defined by

$$(P, Q) := \sum_{t \in \mathbb{N}_T} \sum_{|i| \leq s} H^{a(i,k)}(\mathrm{pr}_2^* h, \nabla_0)(P_{i,t}, Q_{i,t})$$

where h is any scalar product on $\mathbb{R}^{N^2}$ (any two choices yield topologically equivalent scalar products), ∇_0 is the trivial connection. There are more invariant ways to define a scalar product of this kind, but the above definition will fully suffice for our purposes. We will often write S_k^s instead of $S_k^s(\pi)$. By definition, we have $S_{k+1}^s \subset S_k^s$. Obviously, if π_1 is a metric $H^{k(1)}$-vector bundle with connection and π_2 is a metric $H^{k(2)}$-vector bundle with connection then $\pi_1 \oplus \pi_2$ and $\pi_1 \otimes \pi_2$ with the corresponding natural metrics and connections are metric H^k-vector bundles with connections, for $k := \min\{k_1, k_2\}$. To make an analogous statement on the dual vector bundle, recall that on a compact manifold standard estimates (see e.g. [6], VI.3 (p. 386)) give $W^{k,p} \cdot W^{l,q} \subset W^{m,p}$ for all $p, q \in \mathbb{R}$ with $1 < p \leq q \leq \infty$ and for all $k, l, m \in \mathbb{N}$ with $k + l > m + n/q$ and $k, l \geq m$. Specializing to Sobolev Hilbert spaces ($p = q = 2$), this means $H^k \cdot H^l \subset H^{S(k,l)}$ for

$$S(k, l) := \min\{k, l, k + l - \lfloor n/2 \rfloor - 1\}, \tag{1}$$

as long as $k + l - \lfloor n/2 \rfloor - 1 \geq 0$.

Theorem 1. *Let P be a differential operator of order s on a vector bundle $\pi : E \to M$, let $l > n/2$.*

1. *$P^{(l)} := P|_{H^l(\pi)} : H^l(\pi) \to H^{l-s}(\pi)$ is bounded if and only if P is l-safe, and $e^{(l)} : P \mapsto P^{(l)}$ is a smooth linear map from $S_l^s(\pi)$ to the space of bounded linear maps from $H^l(\pi)$ to $H^{l-s}(\pi)$.*
2. *If $k > n/2$ and if π is equipped with a vector bundle metric g of regularity H^k and M has a volume form ω of regularity H^{k-s}, then for $k - s > n/2$, taking the formal $H^0(g, \omega)$-adjoint $P'_{g,\omega}$ of a k-safe operator P of order s is a bounded map $S_k^s(\pi) \to S_{k-s}^s(\pi)$.*
3. *If $k > n/2$ and $s_1 + s_2 \leq k$, then composition of operators is a bounded bilinear map $S_k^{s_1}(\pi) \times S_k^{s_2}(\pi) \to S_k^{s_1+s_2}(\pi)$.*

Proof. The first assertion can be verified by simply counting orders. The necessity of the condition can be seen by the classical fact of sharpness of $S(u, v)$ in Eq. (1) (which can be verified in an open ball in $\mathbb{R}^n$ considering the functions $x \mapsto |x|^\alpha$, for different values of α) combined with appropriate partitions of unity.

Now the formal $H^0(g)$-adjoint $P'_{g,\omega}$ of an operator $P := \sum_{|j| \leq s} P_j \partial_j$ (by linearity w.l.o.g. applied to a section u supported in a single trivialization chart) is given as

$$(P'_{g,\omega})u(x) = \sum_{|j|<s} (-1)^{|j|} \partial_j (P_j^g(x) w(x) u(x)),$$

where P^g denotes the transpose of a matrix with respect to g and $\omega = w dx_1 \wedge \cdots \wedge dx_n$ in the local coordinates. Thus for the coefficients of $P'_{g,\omega}$ we get

$$(P'_{g,\omega})_j = \sum_{l \geq j} (-1)^{|j|} \binom{l}{j} \partial_{l-j} (P_l^g \cdot w),$$

where we used subtraction of multiindices. If P is k-safe, then $P_l, P_l^g \in H^{a(l,k)}$, and $P'_{g,\omega}$ is seen to be $(k-s)$-safe by counting orders. It is easy to see that the map is continuous.