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NON-NATURALLY REDUCTIVE EINSTEIN METRICS ON EXCEPTIONAL LIE GROUPS

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ABSTRACT. Given an exceptional compact simple Lie group G we describe new left-invariant Einstein metrics which are not naturally reductive. In particular, we consider fibrations of G over flag manifolds with a certain kind of isotropy representation and we construct the Einstein equation with respect to the induced left-invariant metrics. Then we apply a technique based on Gröbner bases and classify the real solutions of the associated algebraic systems. For the Lie group G_2 we obtain the first known example of a left-invariant Einstein metric, which is not naturally reductive. Moreover, for the Lie groups E_7 and E_8, we conclude that there exist non-isometric non-naturally reductive Einstein metrics, which are $\mathrm{Ad}(K)$-invariant by different Lie subgroups K.

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1. INTRODUCTION

In 1979 D'Atri and Ziller [DZ] studied naturally reductive metrics on compact semi-simple Lie groups. They gave a complete classification of such metrics on compact simple Lie groups and described many naturally reductive Einstein metrics. They also asked the following question (cf. [DZ] Remark p. 62):

Question. *Given a compact simple Lie group G, do there exist left-invariant Einstein metrics which are not naturally reductive?*

The first left-invariant Einstein metrics on a compact simple Lie group which are non-naturally reductive, were discovered for SU_n ($n \geq 6$) by K. Mori in 1994 [M]. He considered the Lie group SU_n as a principal bundle over the generalized flag manifold $\mathrm{SU}_n / \mathrm{S}(\mathrm{U}_\ell \times \mathrm{U}_m \times \mathrm{U}_k)$ ($\ell + m + k = n \geq 2$) and then he used the reverse of Kaluza-Klein ansatz to describe new left-invariant Einstein metrics. In 2008, Arvanitoyeorgos, Mori and the second author proved the existence of new non-naturally reductive Einstein metrics for SO_n ($n \geq 11$), Sp_n ($n \geq 3$), E_6, E_7 and E_8, using fibrations of a compact simple Lie group over a flag manifold (Kähler C-space) with two isotropy summands (see [AMS]). More recently, Chen and Liang [CL] proved that there is a non-naturally reductive left-invariant Einstein metric on the exceptional Lie group F_4.

In this paper we describe new non-naturally reductive Einstein metrics on compact simple Lie groups G, which can be viewed as principal bundles over flag manifolds $M = G/K$ with three isotropy summands and second Betti number $b_2(M) = 1$. Hence, the painted Dynkin diagram of M is defined by a pair (Π, Π_K) such that $\Pi \backslash \Pi_K = \{\alpha_{i_o}\}$ with $\mathrm{ht}(\alpha_{i_o}) = 3$ for some simple root α_{i_o}. Here, $\Pi = \{\alpha_1, \ldots, \alpha_\ell\}$ is a basis of simple roots and $\mathrm{ht}(\alpha_j)$ is the height (Dynkin mark) of a simple root α_j. Because the heights of a classical compact simple Lie group are bounded by $1 \leq \mathrm{ht}(\alpha_i) \leq 2$ for any $i = 1, \ldots, \ell$ (c.f. [GOV]), the examined Lie groups are necessarily *exceptional*, see Table 2. From now on, we shall denote such a Lie group G by $G(\alpha_{i_o}) \equiv G(i_o)$; then one can immediately encode the isotropy subgroup K via the corresponding painted Dynkin diagram. Moreover, the related reductive decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ induces the left-invariant metrics $\langle\ ,\ \rangle$ that we are interested in.

By extending the notation of [AMS], and since in our case the painted black simple root α_{i_o} is never connected with the vertex corresponding to the negative of the maximal root $\tilde{\alpha} := \mathrm{ht}(\alpha_1)\alpha_1 + \ldots + \mathrm{ht}(\alpha_\ell)\alpha_\ell$, we agree to say that $G \equiv G(i_o)$ is of Type I_b, II_b, or III_b, if after deleting the black vertex the Dynkin diagram splits into *one*, *two*, or *three* components (subdiagrams), respectively. In [AMS] and for compact simple Lie groups G associated to flag manifolds $M = G/K$ with *two* isotropy summands, it was shown that the *new* non-naturally reductive Einstein metrics appear only for the corresponding classes of Type I_b and II_b (for the Types I_a, II_a, III_a the painted black simple root is connected to $-\tilde{\alpha}$). In particular, for such flag manifolds, there are still Lie groups of Types III_a, III_b (related to $\mathrm{SO}_{2\ell}$, see Theorem 3.3) but these cases have not been examined yet. In this study, we focus on exceptional flag manifolds and provide the existence of *new* left-invariant non-naturally reductive Einstein metrics on simple Lie groups of all 3 types I_b, II_b and III_b.