Journal of Geometry and Physics 116 (2017) 228–243

Contents lists available at ScienceDirect

Journal of Geometry and Physics

journal homepage: www.elsevier.com/locate/jgp

Multiscale method, central extensions and a generalized Craik–Leibovich equation

Cheng Yang *

School of Mathematical Sciences, Fudan University, Shanghai 200433, People's Republic of China
Department of Mathematics, University of Toronto, Toronto, ON M5S 2E4, Canada

ARTICLE INFO

Article history:
Received 29 June 2016
Accepted 5 February 2017
Available online 14 February 2017

Keywords:
Multiscale method
Central extensions
Craik–Leibovich equation
Averaging theorem
Adiabatic invariants

ABSTRACT

In this paper we develop perturbation theory on the reduced space of a principal G-bundle. This theory uses a multiscale method and is related to vibrodynamics. For a fast oscillating motion with the symmetry Lie group G, we prove that the averaged equation (i.e. the equation describing the slow motion) is the Euler equation on the dual of a certain central extension of the corresponding Lie algebra $\mathfrak{g}$. As an application of this theory we study the Craik–Leibovich (CL) equation in hydrodynamics. We show that CL equation can be regarded as the Euler equation on the dual of an appropriate central extension of the Lie algebra of divergence-free vector fields. From this geometric point of view, one can give a generalization of CL equation on any Riemannian manifold with boundary.

For accuracy of the averaged equation, we prove that the difference between the solution of the averaged equation and the solution of the perturbed equation remains small (of order ϵ) over a very long time interval (of order $1/\epsilon^2$). Combining the geometric structure of the generalized CL equation and the averaging theorem, we present a large class of adiabatic invariants for the perturbation model of the Langmuir circulation in the ocean.

1. Introduction

In this paper we develop perturbation theory to study a fast oscillating system corresponding to a Lie group G. More specifically, we apply a fast–slow multiscale method to an oscillating Hamiltonian system on the reduced space of a principal G-bundle. It turns out that the averaged equation describing the slow motion is the Euler equation on the dual of the central extension of the corresponding Lie algebra $\mathfrak{g}$.

We hope this theory can shed some light on the geometric nature of some famous equations in mathematical physics, e.g. Craik–Leibovich equation for Langmuir circulation in oceans, infinite conductivity equation in plasma physics, β-plane or Rossby waves equation for a rotating fluid.

This perturbation theory is related to vibrodynamics, an area of dynamics and hydrodynamics studying the behaviour of mechanical and fluid systems subject to fast oscillations. An interesting related example is the stability of the upper position of a pendulum with a vibrating suspension point. Vibrodynamics was studied by many authors including Kapitza, Landau, Bogolyubov, Yudovich, etc. A generalized Krylov–Bogolyubov averaging method related to two-timing procedure was studied by Yudovich, Vladimirov, etc. and is a major tool in this area.

* Correspondence to: School of Mathematical Sciences, Fudan University, Shanghai 200433, People's Republic of China.
E-mail addresses: chyang@math.toronto.edu, c_yang11@fudan.edu.cn.

http://dx.doi.org/10.1016/j.geomphys.2017.02.004

Another interesting example in vibrodynamics is a flow with a fast oscillation related to the boundary conditions. By studying the oscillating flow, one can derive the Craik–Leibovich (CL) equation describing the Langmuir circulation in oceans. Recall that the CL equation is

$$\frac{\partial v}{\partial t} + (v, \nabla)v + curl\ v \times V_0 = -\nabla p, \tag{1.1}$$

where V_0 is a prescribed Stokes drift velocity. This equation was first studied by Craik and Leibovich in [1]. Vladimirov and his coauthors give a new derivation of Craik–Leibovich equation by using the generalized Krylov–Bogolyubov averaging method related to two-timing method, see [2].

In this paper we generalize this two-timing/averaging method to a perturbation theory on the principal G-bundle. By applying this theory to a principal $SDiff(D)$-bundle considered in [3] to describe the free boundary fluid motion, we derive the CL equation. This theory also leads us to the geometric meaning of the CL equation: it turns out to be the Euler equation on the dual of a certain central extension of the Lie algebra of divergence-free vector fields. This geometric point of view enables us to give a higher-dimensional generalization of the CL equation on any Riemannian manifold with boundary in any dimension. Also, a large class of invariant functionals follows from this geometric structure.

Euler equations on the duals of central extensions of Lie algebras arise in many interesting settings in mathematical physics.

Example 1.1. In [4] Khesin and Chekanov studied the infinite conductivity equation on a Riemannian manifold M:

$$\frac{\partial v}{\partial t} = -(v, \nabla)v - v \times B - \nabla p, \tag{1.2}$$

where B is a constant divergence-free magnetic field. This equation is the Euler equation on the dual space of the central extension of the Lie algebra of the divergence-free vector fields $SVect(M)$. The corresponding 2-cocycle is a Lichnerowicz 2-cocycle (see Section 3.2) related to the magnetic field B:

$$\widehat{\omega}_B(X, Y) = \int_M i_X i_Y i_B \nu,$$

where B is an $(n-2)$-vector field corresponding to a closed 2-form on M. Khesin and Chekanov generalized the infinite conductivity equation to any Riemannian manifolds in any dimension and found a large class of invariant functionals.

Remark 1.1. Since the CL equation has a geometric structure similar to the infinite conductivity equation, we are able to prove that those invariants for the infinite conductivity equation turn out to be also invariants for the CL equation. This also helps to construct a large class of adiabatic invariants for the fast–slow system related to the CL equation.

Example 1.2. In [5] Zeitlin studied the β-plane equation (or Rossby waves equation):

$$\dot{\omega} + \{\psi, \omega\} + \beta\psi_x = 0,$$

where β is a constant related to the Coriolis force, ω and ψ are the vorticity and stream functions, respectively. This equation describes the fluid motion on a rotating surface. It is the Euler equation on the dual of a central extension of the Lie algebra of the symplectomorphism group.

Besides the geometric structure, the accuracy of the averaged equation is also considered in this paper. We prove the averaging theorem in a general setting. The averaging theorem combined with the geometric structure of the CL equation enables us to present a class of adiabatic invariants for the fast–slow system related to the CL equation.

Organization and main results of the paper

In Section 2, we give the general setting of the perturbation theory. We derive the averaged equation for a perturbed ODE related to a bilinear operator on a Banach space. Here the main statement is the following averaging theorem:

Theorem 1.1 (=Theorem 2.1)**.** *The difference between the solution of the averaged equation and the solution of the perturbed equation remains small (of order ϵ) over a very long time interval (of order $\frac{1}{\epsilon^2}$).*

In Section 3, we give some preliminaries about an Euler equation on the dual of a Lie algebra and a central extension of a Lie algebra.

In Section 4, we present a general theory on the reduced space of a principal G-bundle. We consider a natural fast–slow Hamiltonian system and derive the averaged equation. The Eulerian nature of this averaged equation is proved in Theorem 4.1:

Theorem 1.2 (=Theorem 4.1)**.** *The averaged equation (i.e. the equation describing the slow motion) is the Euler equation on the dual of a central extension of the corresponding Lie algebra.*