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Complex and Lagrangian surfaces of the complex projective plane via Kählerian Killing Spin^c spinors

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ABSTRACT

The complex projective space $\mathbb{C}P^2$ of complex dimension 2 has a Spin^c structure carrying Kählerian Killing spinors. The restriction of one of these Kählerian Killing spinors to a surface M^2 characterizes the isometric immersion of M^2 into $\mathbb{C}P^2$ if the immersion is either Lagrangian or complex.

1. Introduction

A classical problem in Riemannian geometry is to know when a Riemannian manifold (M^n, g) can be isometrically immersed into a fixed Riemannian manifold $(\widetilde{M}^{n+p}, \widetilde{g})$. The case of space forms $\mathbb{R}^{n+1}$, $\mathbb{S}^{n+1}$ and $\mathbb{H}^{n+1}$ is well-known. In fact, the Gauss, Codazzi and Ricci equations are necessary and sufficient conditions. In other ambient spaces, the Gauss, Codazzi and Ricci equations are necessary but not sufficient in general. Some additional conditions may be required like for the case of complex space forms, products, warped products or 3-dimensional homogeneous spaces (see [1–6]).

In low dimensions, especially for surfaces, another necessary and sufficient condition is now well-known, namely the existence of a special spinor field called *generalized Killing spinor field* [7–10]. These results are the geometrical invariant versions of previous works on the spinorial Weierstrass representation by R. Kusner and N. Schmidt, B. Konopelchenko, I. Taimanov and many others (see [11–13]). This representation was expressed by T. Friedrich [7] for surfaces in $\mathbb{R}^3$ and then extended to other 3-dimensional (pseudo-)Riemannian manifolds [8,6,14,15] as well as for hypersurfaces of 4-dimensional space forms and products [10] or hypersurfaces of 2-dimensional complex space forms by means of Spin^c spinors [16].

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More precisely, the restriction φ of a parallel spinor field on $\mathbb{R}^{n+1}$ to an oriented Riemannian hypersurface M^n is a solution of the generalized Killing equation

$$\nabla_X\varphi = -\frac{1}{2}A(X)\cdot\varphi, \tag{1}$$

where "$\cdot$" and ∇ are respectively the Clifford multiplication and the spin connection on M^n, the tensor A is the Weingarten tensor of the immersion and X any vector field on M. Conversely, T. Friedrich proved in [7] that, in the two dimensional case, if there exists a generalized Killing spinor field satisfying Eq. (1), where A is an arbitrary field of symmetric endomorphisms of TM, then A satisfies the fundamental Codazzi and Gauss equations in the theory of embedded hypersurfaces in a Euclidean space and consequently, A is the Weingarten tensor of a local isometric immersion of M into $\mathbb{R}^3$. Moreover, in this case, the solution φ of the generalized Killing equation is equivalently a solution of the Dirac equation

$$D\varphi = H\varphi, \tag{2}$$

where D denotes the Dirac operator on M, $|\varphi|$ is constant and H is a real-valued function (which is the mean curvature of the immersion in $\mathbb{R}^3$).

More recently, this approach was adapted by the second author, P. Bayard and M.A. Lawn in codimension two, namely, for surfaces in Riemannian 4-dimensional real space forms [17], and then generalized in the pseudo-Riemannian setting [18,19] as well as for 4-dimensional products [20]. As pointed out in [21], this approach coincides with the Weierstrass type representation for surfaces in $\mathbb{R}^4$ introduced by Konopelchenko and Taimanov [12,22].

The aim of the present article is to provide an analogue for the complex projective space $\mathbb{C}P^2$. The key point is that, contrary to the case of hypersurfaces of $\mathbb{C}P^2$ which was considered in [16], the use of Spin^c parallel spinors is not sufficient. Indeed, for both canonical or anti-canonical Spin^c structures, the parallel spinors are always in the positive half-part of the spinor bundle. But, as proved in [17] or [20], a spinor with non-vanishing positive and negative parts is required to get the integrability condition of an immersion in the desired target space. For this reason, Spin^c parallel spinors are not adapted to our problem. Thus, we make use of real Kählerian Killing spinors. Therefore, our argument holds for the complex projective space and not for the complex hyperbolic space, since $\mathbb{C}\mathbb{H}^2$ does not carry a real or imaginary Kählerian Killing spinor.

We will focus on the case of complex and Lagrangian immersions into $\mathbb{C}P^2$. These two cases and especially the Lagrangian case are of particular interest in the study of surfaces in $\mathbb{C}P^2$ (see [23–25] and references therein for instance).

First, consider (M^2, g) an oriented Riemannian Spin^c surface and E an oriented Spin^c vector bundle of rank 2 over M with scalar product $\langle\cdot,\cdot\rangle_E$ and compatible connection ∇^E. We denote by F^M (resp. F^E) the curvature form (an imaginary 2-form on M) of the auxiliary line bundle defining the Spin^c structure on M (resp. on the vector bundle E). For a spinor field φ, we define $\bar\varphi$ by $\bar\varphi = \varphi_+ - \varphi_-$, where φ^+ and φ^- denote the positive and negative half parts of φ (see Section 2). They are the projections of φ on the eigensubspaces for the eigenvalues $+1$ and -1 of the complex volume form. The aim of the paper is to prove the following two results:

The first theorem gives a spinorial characterization of complex immersions of surfaces in the complex projective space $\mathbb{C}P^2$.

Theorem 1.1. *Let (M^2, g) be an oriented Riemannian surface and E an oriented vector bundle of rank 2 over M with scalar product $\langle\cdot,\cdot\rangle_E$ and compatible connection ∇^E. We denote by $\Sigma = \Sigma M\otimes\Sigma E$ the twisted spinor bundle. Let $B : TM\times TM \longrightarrow E$ be a bilinear symmetric map, $j : TM \longrightarrow TM$ a complex structure on M and $t : E \longrightarrow E$ a complex structure on E. Assume moreover that $t(B(X, Y)) = B(X, j(Y))$ for all $X \in \Gamma(TM)$ and consider $\{e_1, e_2\}$ an orthonormal frame of TM. Then, the following two statements are equivalent*

(1) *There exist a Spin^c structure on $\Sigma M\otimes\Sigma E$ whose auxiliary line bundle's curvature is given by $F^{M+E}(e_1, e_2) := F^M(e_1, e_2) + F^E(e_1, e_2) = 0$ and a spinor field $\varphi \in \Gamma(\Sigma M\otimes\Sigma E)$ satisfying for all $X \in \Gamma(TM)$,*

$$\nabla_X\varphi = -\frac{1}{2}\eta(X)\cdot\varphi - \frac{1}{2}X\cdot\varphi + \frac{i}{2}j(X)\cdot\overline{\varphi}, \tag{3}$$

such that φ^+ and φ^- never vanish and where η is given by

$$\eta(X) = \sum_{j=1}^{2} e_j\cdot B(e_j, X).$$

(2) *There exists a local isometric **complex** immersion of (M^2, g) into $\mathbb{C}P^2$ with E as normal bundle and second fundamental form B such that the complex structure of $\mathbb{C}P^2$ over M is given by j and t (in the sense of Proposition 3.2).*

The second theorem is the analogue of Theorem 1.1 for Lagrangian surfaces in $\mathbb{C}P^2$.

Theorem 1.2. *Let (M^2, g) be an oriented Riemannian surface and E an oriented vector bundle of rank 2 over M with scalar product $\langle\cdot,\cdot\rangle_E$ and compatible connection ∇^E. We denote by $\Sigma = \Sigma M\otimes\Sigma E$ the twisted spinor bundle. Let $B : TM\times TM \longrightarrow E$ be a bilinear symmetric map, $h : TM \longrightarrow E$ a bundle map and $s : E \longrightarrow TM$ the dual map of h. Assume moreover that h and s are parallel, $h\circ s = -\mathrm{id}_E$ and $A_{h(Y)}X + s(B(X, Y)) = 0$, for all $X \in \Gamma(TM)$, where $A_\nu : TM \longrightarrow TM$ is defined by $g(A_\nu X, Y) = \langle B(X, Y), \nu\rangle_E$ for all $X, Y \in \Gamma(TM)$ and $\nu \in E$. Then, the following two statements are equivalent*