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Lifting a weak Poisson bracket to the algebra of forms

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ABSTRACT

We detail the construction of a weak Poisson bracket over a submanifold Σ of a smooth manifold M with respect to a local foliation of this submanifold. Such a bracket satisfies a weak type Jacobi identity but may be viewed as a usual Poisson bracket on the space of leaves of the foliation. We then lift this weak Poisson bracket to a weak odd Poisson bracket on the odd tangent bundle ΠTM, interpreted as a weak Koszul bracket on differential forms on M. This lift is achieved by encoding the weak Poisson structure into a homotopy Poisson structure on an extended manifold, and lifting the Hamiltonian function that generates this structure. Such a construction has direct physical interpretation. For a generic gauge system, the submanifold Σ may be viewed as a stationary surface or a constraint surface, with the foliation given by the foliation of the gauge orbits. Through this interpretation, the lift of the weak Poisson structure is simply a lift of the action generating the corresponding BRST operator of the system.

1. Introduction and background

When considering classical gauge systems, one usually starts with a Lagrangian or Hamiltonian function which may then be used to derive the equations of motion through the least action principle. In most cases this function is known, but in those that are not, it is known that the existence of a classical BRST differential allows one to identify the equations of motion and the gauge symmetries independently of the Lagrangian. Such a BRST differential is a homological vector field on an appropriately extended manifold which encodes the gauge system [1]. In [2], a general geometric set-up for an arbitrary gauge system was described, and an embedding of such a system into an extended manifold was constructed. This is without reference to any Lagrangian or Hamiltonian function and merely assumes the existence of the equations of motion, together with the gauge symmetries present. This geometric construction introduced a weak Poisson bracket, a Poisson bracket that satisfies a weak Jacobi identity, which allows a quantisation of these gauge systems which may not be Lagrangian or Hamiltonian. This paper is about lifting this weak Poisson bracket to the algebra of differential forms by utilising the described embedding into the extended manifold.

For an arbitrary smooth manifold M with local coordinates (x^i), let the system of equations $T^a(x) = 0$, $a = 1, \dots, k$, define a smooth submanifold Σ of codimension k. Choose n linearly independent vector fields R_α on M such that they are tangent to Σ. (In fact, linear independence is not necessary but simplifies the exposition, see Remark 1.) For a corresponding gauge system in the Lagrangian formalism, the equations $T^a(x) = 0$ may be identified with the equations of motion, possibly derived from an action principle, whilst the manifold M is understood as the space of trajectories in a configuration space of the system. The vector fields R_α then generate the gauge symmetries of the action. If we consider the Hamiltonian formalism,

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then the surface Σ may be identified with the constraint surface given by the constraint equations $T^a(x) = 0$. The vector fields correspond to the gauge generators which define a foliation of the submanifold Σ into gauge orbits identified with the integral submanifolds.

To obtain this foliation, it is required for the vector fields to form an integrable distribution over Σ; they must satisfy the commutation relation

$$[R_\alpha, R_\beta] = f^\gamma_{\alpha\beta} R_\gamma + T^a X_{a\alpha\beta}, \tag{1}$$

for smooth functions $f^\gamma_{\alpha\beta}$ and vector fields $X_{a\alpha\beta}$ on M. Notice that the presence of the constraint terms T^a means that this is an open Lie algebra over M that closes only over Σ. The space of leaves N of the foliation of Σ generated by this integrable distribution gives the true physical degrees of freedom when viewed as a gauge system. Because of this, physically interesting objects are those that descend to the leaf space, i.e. those which are constant over the gauge orbits which are identified with the integral submanifolds. In the work [2], a function or multivector field was defined to be projectible precisely when it may be considered as a smooth contravariant tensor field on the leaf space N. In general, the space N may not be smooth. When we say smooth in this sense, we refer to those smooth tensor fields $\mathcal{T}$ on M which are constant along the integral submanifolds:

$$\mathcal{L}_{R_\alpha} \mathcal{T} = \mathcal{T}^\alpha R_\alpha + T^a \mathcal{T}_a,$$

for smooth tensor fields $\mathcal{T}_a, \mathcal{T}^\alpha$; here $\mathcal{L}_{R_\alpha}\mathcal{T}$ is the Lie derivative along the vector field R_α. Therefore, we should consider projectible multivector fields as physically interesting. Indeed, examples include: vector fields R_α that correspond to the gauge generators, vector fields that introduce dynamics into the gauge system, and bivector fields which can induce Poisson structures in the algebra of functions of N. (In physical literature the algebra of functions on N, identified with projectible functions on M, is called the algebra of physical observables.) Such a Poisson structure on N, specified by an appropriate projectible bivector field on M, induces a weak Poisson bracket on M. This is a Poisson bracket on M corresponding to the same projectible bivector field, but which satisfies only a weak Jacobi identity; a Jacobi identity that holds over Σ up to terms proportional to the vector fields R_α.

An embedding of this foliation together with a weak Poisson bracket was detailed, [2], into which the information was encoded into a single function S on an extended manifold. This embedding corresponds to reformulating the theory in terms of the BRST language [1]. The function S is called the master function and corresponds to the action associated to the gauge system. Such a function contains all the information present in the gauge system, and generates the BRST operator, a homological vector field on the extended manifold. This extended manifold is the original manifold M appropriately extended by ghost variables. The notion of projectibility was then entwined with the BRST operator of the theory, where it was shown that projectible multivector fields are cocycles of this BRST operator considered as a differential on the larger algebra of functions.

The BRST operator may be considered as a unary bracket in a homotopy Poisson structure (a P_∞-structure), which is generated by the master function S on the extended manifold. The weak Poisson bracket on M embeds into this homotopy Poisson structure, and is recovered as the restriction of the binary bracket to M. In the work [3] an explicit construction was given to canonically lift a homotopy Poisson structure to an odd homotopy Poisson structure (an S_∞-structure) on the odd tangent bundle. By applying this construction, the even homotopy Poisson structure may be lifted to the odd tangent bundle of the extended manifold, to produce the corresponding odd homotopy structure. Through this we may define a weak Koszul bracket on the odd tangent bundle ΠTM. This weak bracket is the restriction of the binary bracket in this odd homotopy Poisson structure, and corresponds to the weak Poisson bracket on M in precisely the same way that the well-known Koszul bracket of forms corresponds to a Poisson structure.

The Koszul bracket is a natural odd extension of the usual even Poisson bracket. A natural even extension to the entire algebra of forms does not exist; however, in the works [4,5], it was shown that a Poisson bracket induces a genuine even Poisson bracket in the space of co-exact forms—differential forms modulo the exact forms. The exterior differential d from co-exact forms into differential forms was shown to be a homomorphism of Lie algebras, taking the even Poisson bracket on co-exact forms to the odd Koszul bracket on exact forms. The exact forms in this case form an ideal in the graded Lie algebra of forms endowed with the Koszul bracket.

The extension of the algebra of physical observables by differential forms now implies a proper generalisation of the notion of a physical state. Regarding the usual classical states, the points of a phase space, as 0-cycles in the sense of algebraic topology, it is natural to consider the cycles of higher degrees represented by higher-dimensional closed surfaces, possibly with singularities. These may be viewed as sort of mixed states in classical mechanics, when only part of the physical data is exactly known about the system. Integration of co-exact forms over cycles then yields the measured values of physical observables in such mixed states. From this perspective, the construction of the weak Koszul bracket on forms being proposed in this paper, extends the previous results of [4,5] to the case of constrained and gauge invariant dynamical systems. More precisely, it may be shown that each weak Koszul bracket induces a genuine Poisson bracket in the space of projectible co-exact forms, and the dynamics of form-valued physical observables are governed by a projectible vector field compatible with the Poisson bracket. We will detail this construction elsewhere.

In Sections 2 and 3 we review the construction introduced in the work [2]. We recall the weak Poisson bracket and the embedding of the geometrical set-up into the extended manifold. After, we discuss the relationship between projectible multivector fields and the homological vector field, the BRST operator of an associated gauge system. In Section 4 the