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Killing tensors on tori

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ABSTRACT

We show that Killing tensors on conformally flat n-dimensional tori whose conformal factor only depends on one variable, are polynomials in the metric and in the Killing vector fields. In other words, every first integral of the geodesic flow polynomial in the momenta on the sphere bundle of such a torus is linear in the momenta.

1. Introduction

Killing tensors are symmetric p-tensors with vanishing symmetrized covariant derivative and correspond to Killing vector fields for $p = 1$. Originally, Killing tensors were studied in the physics literature since they define first integrals (polynomial in the momenta) of the equation of motion, and thus functions constant on geodesics. This property makes Killing tensors very important in the theory of integrable systems.

First integrals of the geodesic flow on the 2-dimensional torus is an intensively studied topic. It is easy to see that every metric $\tilde{g}$ on T^2 with linear first integrals, i.e. Killing vector fields, is of the form $\tilde{g} = e^{2f(ax+by)}(\mathrm{d}x^2 + \mathrm{d}y^2)$, where $a, b \in \mathbb{R}$ and f is a periodic real function [1, Thm. 8]. Indeed, by the uniformization theorem, every metric on T^2 is conformal to a flat metric, i.e. $\tilde{g} = e^{2f(x,y)}(\mathrm{d}x^2 + \mathrm{d}y^2)$ where f is invariant by a lattice in $\mathbb{R}^2$. A Killing vector field ξ with respect to $\tilde{g}$ is conformal Killing with respect to the flat metric $g = \mathrm{d}x^2 + \mathrm{d}y^2$, thus ∇^g-parallel by [2, Prop. 6.6]. This shows that there exist real constants a, b such that $\xi = b\frac{\partial}{\partial x} - a\frac{\partial}{\partial y}$. As ξ preserves both g and $\tilde{g}$, one has $\xi(f) = 0$, so $f(x, y)$ is a function of $ax + by$.

There also is a classification of metrics with quadratic first integrals, i.e. with Killing (non-parallel) 2-tensors. These metrics turn out to be of Liouville type, i.e. in conformal coordinates the metric can be written as $\tilde{g} = (f(x)^2+g(y)^2)(\mathrm{d}x^2+\mathrm{d}y^2)$ cf. [3]. Surprisingly, the existence of first integrals of degree $\geq$ 3 independent of those of degree 1 and 2 on a 2-torus is a completely open problem. The conjecture is that there are no such first integrals cf. [1]. During the last thirty years many partial results were proved supporting this conjecture cf. [4–6].

It follows from the theory of integrable systems that besides the metric, it is not possible to have two other functional independent first integrals on the 2-torus cf. [7]. Indeed, in this situation the geodesic flow would be *superintegrable*. Then

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its trajectories would lie in the intersection of three level sets, one of which is the compact sphere bundle. One can conclude that all geodesics have to be closed. Then the manifold has the homology ring of a rank one symmetric space, which is not the case for the torus.

In particular this means that if the metric $\tilde{g}$ on T^2 carries a Killing vector field, then any Killing tensor of higher degree is functional dependent on it, i.e. it is expressible as polynomial in the Killing vector field and the metric. Since every metric $\tilde{g}$ on T^2 is conformal to the flat metric g, and has a Killing vector field if and only if the conformal factor only depends on one coordinate of T^2, the above fact can be equivalently stated as follows: *Every Killing tensor on the torus T^2 equipped with a metric of the form $\tilde{g} = e^{2f(y)}(\mathrm{d}x^2 + \mathrm{d}y^2)$ is a polynomial in the Killing vector field $\frac{\partial}{\partial x}$ and in the metric tensor $\tilde{g}$.*

In our article we generalize this fact to the n-dimensional torus. Our main result is

Theorem 1.1. *Let $f : \mathbb{R} \to \mathbb{R}$ be a non-constant 2π-periodic function and let K be a Killing tensor on the torus T^n equipped with the metric $\tilde{g} = e^{2f(x_n)}(\mathrm{d}x_1^2 + \cdots + \mathrm{d}x_n^2)$. Then K is a polynomial in the Killing vector fields $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_{n-1}}$ and in the metric tensor $\tilde{g}$.*

The idea of the proof is as follows. We first translate the Killing equation on the flat torus. Next, using the formalism developed in [2], we show that the components of any Killing tensor with respect to the flat metric are constant functions in $x_1, \ldots, x_{n-1}$. The Killing equation then reduces to a system of ordinary differential equations with polynomial solutions, which translated back to the metric $\tilde{g}$ yields the result.

Note that for dimensional reasons, the result above cannot be proved by the above arguments from the theory of integrable systems for $n \geq 3$. Indeed, assuming that K is a Killing tensor on $(T^n, \tilde{g})$, functionally independent of the Killing vector fields $\xi_1, \ldots, \xi_{n-1}$ and on the metric tensor $\tilde{g}$, then one would obtain $n + 1$ first integrals of the geodesic flow on the tangent bundle of $(T^n, \tilde{g})$, but this no longer implies superintegrability since $n + 1 < 2n - 1$ for $n \geq 3$.

2. Preliminaries

We will use the formalism introduced in our article [2]. For the convenience of the reader, we recall here the standard definitions and formulas which are relevant in the sequel.

Let $(\mathrm{T}M, g)$ be the tangent bundle of a n-dimensional Riemannian manifold (M, g). We denote with $\mathrm{Sym}^p\,\mathrm{T}M \subset \mathrm{T}M^{\otimes p}$ the p-fold symmetric tensor product of $\mathrm{T}M$. The elements of $\mathrm{Sym}^p\,\mathrm{T}M$ are linear combinations of symmetrized tensor products

$$v_1 \cdot \cdots \cdot v_p := \sum_{\sigma \in S_p} v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(p)}\,,$$

where $v_1, \ldots, v_p$ are tangent vectors in $\mathrm{T}M$.

Let $\{e_i\}$ denote from now on a local orthonormal frame of $(\mathrm{T}M, g)$. Using the metric g, we will identify $\mathrm{T}M$ with T^*M and thus $\mathrm{Sym}^2\mathrm{T}^*M \simeq \mathrm{Sym}^2\,\mathrm{T}M$. Under this identification we view the metric tensor as a symmetric 2-tensor $\mathrm{L} := 2g = \sum e_i \cdot e_i$. The scalar product g induces a scalar product, also denoted by g, on $\mathrm{Sym}^p\,\mathrm{T}M$ defined by

$$g(v_1 \cdot \cdots \cdot v_p, w_1 \cdot \cdots \cdot w_p) \;:=\; \sum_{\sigma \in S_p} g(v_1, w_{\sigma(1)}) \cdot \cdots \cdot g(v_p, w_{\sigma(p)}).$$

Using this scalar product, every element K of $\mathrm{Sym}^p\,\mathrm{T}M$ can be identified with a polynomial map of degree p on $\mathrm{T}M$, defined by the formula $K(v_1, \ldots, v_p) = g(K, v_1 \cdot \ldots \cdot v_p)$. The metric adjoint of the bundle homomorphism $v\cdot : \mathrm{Sym}^p\,\mathrm{T}M \to \mathrm{Sym}^{p+1}\,\mathrm{T}M,\ K \mapsto v \cdot K$ is the contraction map $v \lrcorner : \mathrm{Sym}^{p+1}\,\mathrm{T}M \to \mathrm{Sym}^p\,\mathrm{T}M,\ K \mapsto v \lrcorner K$, defined by

$$(v \lrcorner K)(v_1, \ldots, v_{p-1}) := K(v, v_1, \ldots, v_{p-1}).$$

The metric adjoint of $L\cdot : \mathrm{Sym}^p\,\mathrm{T}M \to \mathrm{Sym}^{p+2}\,\mathrm{T}M$ is the bundle homomorphism

$$\Lambda : \mathrm{Sym}^{p+2}\,\mathrm{T}M \to \mathrm{Sym}^p\,\mathrm{T}M, \quad K \mapsto \sum e_i \lrcorner\, e_i \lrcorner K.$$

The following commutator formulas are straightforward:

$$[\,\Lambda,\ v\,\cdot\,] = 2\,v\,\lrcorner, \quad [\,v \lrcorner,\ \mathrm{L}\,] = 2\,v\cdot, \quad [\,\Lambda,\ v \lrcorner\,] = 0 = [\,\mathrm{L},\ v\cdot\,]. \tag{1}$$

For later use, let us state the following formula which holds for any vector $v \in \mathrm{T}M$ and symmetric tensor $K \in \mathrm{Sym}^q(\mathrm{T}M)$:

$$(L \cdot K)(v, \ldots, v) = (q + 2)(q + 1)K(v, \ldots, v)|v|^2. \tag{2}$$

Indeed, using (1) repeatedly we may write

$$(\mathrm{L} \cdot K)(v, \ldots, v) = \langle \mathrm{L} \cdot K, v^{q+2} \rangle = \langle K, \Lambda v^{q+2} \rangle = (q + 2)(q + 1)\langle K, v^q \rangle |v|^2.$$

We denote by $\mathrm{Sym}^p_0\,\mathrm{T}M := \ker(\Lambda : \mathrm{Sym}^p\,\mathrm{T}M \to \mathrm{Sym}^{p-2}\,\mathrm{T}M)$ the space of trace-free symmetric p-tensors. The bundle of symmetric tensors splits as

$$\mathrm{Sym}^p\,\mathrm{T}M \;\cong\; \mathrm{Sym}^p_0\,\mathrm{T}M \;\oplus\; \mathrm{Sym}^{p-2}_0\,\mathrm{T}M \;\oplus\; \ldots\,,$$