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Group invariant transformations for the Klein–Gordon equation in three dimensional flat spaces

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ABSTRACT

We perform the complete symmetry classification of the Klein–Gordon equation in maximal symmetric spacetimes. The central idea is to find all possible potential functions $V(t, x, y)$ that admit Lie and Noether symmetries. This is done by using the relation between the symmetry vectors of the differential equations and the elements of the conformal algebra of the underlying geometry. For some of the potentials, we use the admitted Lie algebras to determine corresponding invariant solutions to the Klein–Gordon equation. An integral part of this analysis is the problem of the classification of Lie and Noether point symmetries of the wave equation.

1. Introduction

The group classification problem was initiated by Ovsiannikov [1] who analyzed the nonlinear heat equation. Since then numerous studies have been devoted to group classifications of fundamental equations that model mathematical, relativistic, biological and physical phenomena [2–8]. In addition, there is now a rich body of literature surrounding Lie symmetry theory, its scheme and vast applications to differential equations [9–13]. In particular, wave and Klein–Gordon equations are of particular interest as they are two important equations in all areas of physics. A knowledge of the Lie symmetry structures of the Klein–Gordon equation in a Riemannian space enables the determination of solutions of this equation which is invariant under a given Lie symmetry.

Indeed, recent investigations [14–20] have revolved around wave, Klein–Gordon, Poisson and Schrödinger equations—showing that the Lie symmetry vectors are obtained directly from the collineations of a metric which defines the underlying geometry in which the evolution occurs. In [21] it was proved that for a linear (in derivatives) second-order partial differential equation (PDE), the Lie point symmetries are related to the conformal algebra of the geometry defined by the PDE. In [22], a geometric approach related the Lie symmetries of the Klein–Gordon equation to the conformal algebra of

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classes of the Bianchi I spacetime and a study of potential functions was performed. Whilst recently the connection between collineations and symmetries was established for a system of quasilinear PDEs [23]. A similar result has been proved for the Poisson equation [24].

In this paper, we use geometric results that transfer the problem of the Lie and Noether symmetry classification of the Klein–Gordon equation to the problem of determining the conformal Killing vectors that admit appropriate potential functions. Inspired from the symmetry classification of the two- and three-dimensional Newtonian systems in which a geometric approach was applied [25,26], in this work in order to perform the classification, the conformal Killing vectors of the space are used to solve a constraint condition. The general results are applied to two practical problems viz., the classification of all potential functions in a three dimensional Euclidean and Minkowski space, for which the Klein–Gordon equation admits Lie and Noether point symmetries and secondly, the Lie point symmetries are used to determine invariant solutions of the equation.

The paper is organized as follows. Section 2 provides the geometrical preliminaries and the theoretical background about symmetry analysis. In Section 3, we state the main theorem containing the constraint condition. Section 4 provides a short review about the spacetime and its properties and we perform the symmetry classification for the Klein–Gordon equation and the corresponding potentials. Section 5 illustrates some invariant solutions for the Klein–Gordon equation using particular potential functions. Finally in Section 6 we draw our conclusions.

2. Preliminaries

In this section we review the definitions and properties of spacetime collineations and of the point symmetries of differential equations.

2.1. Lie and noether point symmetries

Consider a system with q unknown functions u^a which depends on p independent variables x^i, i.e. we denote $u = (u^1, \ldots, u^q)$ and $x = (x^1, \ldots, x^p)$, respectively. Let

$$G_\alpha\left(x, u^{(k)}\right) = 0, \quad \alpha = 1, \ldots, q, \tag{1}$$

be a system of m nonlinear differential equations, where $u^{(k)}$ represents the kth derivative of u with respect to x. A one-parameter Lie group of transformations (ε is the group parameter) that is invariant under (1) is given by

$$\bar{x} = \Xi(x, u; \varepsilon) \quad \bar{u} = \Phi(x, u; \varepsilon). \tag{2}$$

Invariance of (1) under the transformation (2) implies that any solution $u = \Theta(x)$ of (1) maps into another solution $v = \Psi(x; \varepsilon)$ of (1). Expanding (2) around the identity $\varepsilon = 0$, we can generate the following infinitesimal transformations:

$$\begin{aligned} \bar{x}^i &= x^i + \varepsilon\xi^i(x, u) + \mathcal{O}(\varepsilon^2), \quad i = 1, \ldots, p, \\ \bar{u}^\alpha &= u^\alpha + \varepsilon\eta^\alpha(x, u) + \mathcal{O}(\varepsilon^2). \end{aligned} \tag{3}$$

The action of the Lie group can be recovered from that of its infinitesimal generators acting on the space of independent and dependent variables. Hence, we consider the following vector field

$$X = \xi^i\partial_{x^i} + \eta^\alpha\partial_{u^\alpha}. \tag{4}$$

The action of X is extended to all derivatives appearing in the equation in question through the appropriate prolongation. The infinitesimal criterion for invariance is given by

$$X\left[\text{LHS Eq.(1)}\right]|_{Eq.(1)} = 0. \tag{5}$$

Eq. (5) yields an overdetermined system of linear homogeneous equation which can be solved algorithmically, more details can be found in [9] among other texts.

The generalized total differentiation operator D_i with respect to x^i is given by

$$D_i = \frac{\partial}{\partial x^i} + u_i^\alpha\frac{\partial}{\partial u^\alpha} + u_{ij}^\alpha\frac{\partial}{\partial u_j^\alpha} + \cdots \tag{6}$$

and W^α is the characteristic function given by

$$W^\alpha = \eta^\alpha - \xi^j u_j^\alpha. \tag{7}$$

The Euler–Lagrange equations, if they exist, are the system $\delta L/\delta u^\alpha = 0$, where $\delta/\delta u^\alpha$ is the Euler–Lagrange operator given by

$$\frac{\delta}{\delta u^\alpha} = \frac{\partial}{\partial u^\alpha} + \sum_{s\geq 1}(-1)^s D_{i_1}\cdots D_{i_s}\frac{\partial}{\partial u^\alpha_{i_1\cdots i_s}}. \tag{8}$$

L is referred to as a Lagrangian. If we include point dependent gauge terms $f_1, \ldots, f_n$, the Noether symmetries X are given by

$$X(L) + LD_i(\xi^i) = D_i(f_i). \tag{9}$$