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The Clebsch potential approach to fluid Lagrangians

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ABSTRACT

The Clebsch potential approach to fluid lagrangians is developed in order to establish contact with other approaches to fluids. Three variants of the perfect fluid approach are looked at. The first is an explicit linear lagrangian constructed directly from the Clebsch potentials, this has fixed equation of state and explicit expression for the pressure but is less general than a perfect fluid. The second is lagrangians more general than that of a perfect fluid which are constructed from higher powers of the comoving vector. The third is lagrangians depending on two vector fields which can represent both density flow and entropy flow.

1. Introduction

1.1. Motivation

The motivation of this paper is to provided a unified approach to the various ways that fluids are described in physics. In particular the methods used by relativists, fluid mechanists, and nuclear physicists have grown distinct. In many areas of physics a unified approach is provided by the Lagrange method, which for fluids is developed here.

1.2. Methodology

The methodology used is first to simplify a perfect fluid in order to investigate if methods of field theory can be applied to it; and then to generalize a perfect fluid to try and establish contact with more physical fluids.

1.3. Other approaches to fluids

A perfect fluid has a variational formulation [1–4] which uses the first law of thermodynamics. In such a formulation Clebsch potentials [5–10] for the comoving fluid vector field are used. Here this approach is both applied to less general fluids and to more general fluids. Other approaches to fluids include the following **twelve**. The *first* uses lagrangians dependent on combinations of Clebsch potentials which do not necessarily form a vector [11]. The *second* is that the comoving vector can be thought of as $U^a = \dot{x}^a$, so that a perfect fluid is a type of generalization of a point particle, then there turns out to be a fluid generalization of a membrane [12]. The *third* is that the charge substitution $\partial_a \rightarrow \partial_a + \imath e A_a$ can be applied to fluids as well as fields and this leads to a model of symmetry breaking [13]. The *fourth* is that the Navier–Stokes equation has a lagrangian formulation [14–18], but the lagrangian has different measure and also image fields. The *fifth* is that hydrodynamics can be expressed using a grad expansion [19–23] which needs an entropy vector. The *sixth* is that contemporary bjorken models use the grad expansion [24–28]. The *seventh* is fluid plasmas [29,30]. The *eighth* is elastic models [31] §3, where the density

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rather than the pressure is used as the lagrangian. The *ninth* is other quantization methods such as brst and path integral applied to fluids [32]. The *tenth* is superfluids [33]. The *eleventh* is spinning fluids [34–36]. The *twelfth* is cosmology [37], where Clebsch potentials have been used [38–42].

1.4. Conventions

The word potential is disambiguated by referring to potentials for a vector field as Clebsch potentials and potentials that occur in Lagrange theory as coefficient functions. When a measure is suppressed it is $\int\sqrt{-g}dx^4$ not $\int d\tau$ unless otherwise stated. μ is density and $\mathcal{P}$ is the pressure. q is a Clebsch potential. σ is used for a Clebsch potential, a pauli matrix and the shear of a vector, to disambiguate the pauli matrix is always σ_p and the shear labels with which vector it is with respect to $\overset{U}{\sigma}$. Capital Π indicates a momentum with respect to the proper time τ not the coordinate time t. $a, b, c, \ldots$ are spacetime indices, $i, j, k, \ldots$ label sets of fields and momenta, and $\iota, \kappa, \ldots$ label constraints. The signature is $-+++$.

2. The perfect fluid

For a perfect fluid the lagrangian is taken to be the pressure $\mathcal{L} = \mathcal{P}$, and the action is

$$I = \int dx^4 \mathcal{P}. \tag{1}$$

The Clebsch potentials are given by

$$hV_a = W_a = \sigma_a + \theta s_a, \qquad V_a V^a = -1, \tag{2}$$

where if more potentials are needed it is straightforward to instate them; there are several sign conventions for (2). The Clebsch potentials are sometimes given names: σ is called the higgs because it has a similar role to the higgs field in symmetry breaking using fluids [11,13], θ is called the thermasy and s the entropy [3]. Variation is achieved via the first law of thermodynamics

$$\delta\mathcal{P} = n\delta h - nT\delta s = -nV_a\delta W^a - nT\delta s, \qquad nh = \mu + \mathcal{P}, \tag{3}$$

where n is the particle number and h is the enthalpy. Metrical variation yields the stress

$$T_{ab} = (\mu + \mathcal{P})V_a V_b + \mathcal{P}g_{ab}, \tag{4}$$

the nöther currents $j^a = \delta I/\delta q_a$ are

$$j^a_\sigma = -nV^a, \quad j^a_\theta = 0, \quad j^a_s = -n\theta V^a, \tag{5}$$

variation with respect to the Clebsch potentials gives

$$(nV^a)_a = \dot{n} + n\Theta = 0, \quad \dot{s} = 0, \quad \dot{\theta} = T, \tag{6}$$

where $\Theta \equiv V^a_{.;a}$ is the vectors expansion: thus the conservation of the nöther currents (5) gives the same equations (6) as varying the Clebsch potentials; the normalization condition $V_a V^a = -1$ and (6) give

$$\dot{\sigma} = -h. \tag{7}$$

The Bianchi identity is

$$T^{ab}_{..;b} = n\dot{W}^a + \mathcal{P}^{,a}, \tag{8}$$

substituting for W using (2) and for $\mathcal{P}$ using (3) this vanishes identically. If one attempts to apply existing scalar field Fourier oscillator quantization procedures to the above there is the equation

$$W^a_{.;a} = \Box\sigma + \theta_a s^a + \theta\Box s = \left(hV^a\right)_a = \dot{h} + h\Theta = \dot{h} - \frac{\dot{n}}{n}h = h\left(\ln\left(\frac{h}{n}\right)\right)^{\circ}, \tag{9}$$

and if this vanishes the enthalpy n is proportional to the particle number n, for an example of this see Section 3. The pressure $\mathcal{P}$ and density μ are only implicitly defined in terms of the Clebsch potentials so it is not clear what operators should correspond to them. Another possibility is to note that (6) are first order differential equations and to try and replace them with spinorial equations; however this would require a spinorial absolute derivative in place of the vectorial absolute derivative, see [43] §4.4.

The canonical Clebsch momenta are given by $\Pi^i = \delta I/\delta\dot{q}^i$

$$\Pi^\sigma = -n, \quad \Pi^\theta = 0, \quad \Pi^s = -n\theta, \tag{10}$$