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Generalized Lagrangian mean curvature flows in almost Calabi–Yau manifolds

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In this paper, we study the generalized Lagrangian mean curvature flow in almost Einstein manifold proposed by T. Behrndt. We show that the singularity of this flow is characterized by the second fundamental form. We also show that the rescaled flow at a singularity converges to a finite union of Special Lagrangian cones for generalized Lagrangian mean curvature flow with zero-Maslov class in almost Calabi–Yau manifold. As a corollary, there is no finite time Type-I singularity for such a flow.

1. Introduction

Suppose $(M, J, \bar{\omega}, \bar{g})$ is a smooth Kähler manifold with complex dimension n, complex structure J, Kähler metric $\bar{g}$ and Kähler form $\bar{\omega}$. The Kähler form and the Kähler metric are related by

$$\bar{\omega}(X, Y) = \bar{g}(JX, Y),$$

for $X, Y \in \Gamma(TM)$. Moreover, suppose that $\overline{Ric}$ is the Ricci tensor of $\bar{g}$. Then the Ricci form $\bar{\rho}$ is defined by

$$\bar{\rho}(X, Y) = \overline{Ric}(JX, Y).$$

Let L be a compact manifold of real dimension n and $F_0 : L \longrightarrow M$ an immersion of L into M. The induced metric on L is $g = F_0^*\bar{g}$ and set $\omega = F_0^*\bar{\omega}$. It is known by definition that F_0 is a *Lagrangian immersion* if $\omega = 0$.

In 1996, Strominger, Yau and Zaslow [1] found that mirror symmetry is related to special Lagrangian submanifold (which is automatically minimal) in Calabi–Yau manifold. One natural approach to obtaining minimal submanifold is to evolve a submanifold along the negative gradient flow of the area functional, i.e., the mean curvature flow. Fortunately, when the ambient manifold M is Kähler–Einstein, Smoczyk [2] proved that if the initial surface L_0 is Lagrangian, then along the mean curvature flow, it remains Lagrangian for each time. Since then, Lagrangian mean curvature flow received a lot of attention and there are many results on it. (c.f. [3–6], etc.) All of them concern Lagrangian mean curvature flow in Kähler–Einstein manifold, while most of them focus on Calabi–Yau ambient manifold.

Recently, generalized Lagrangian mean curvature flow attracts more attention [7,8]. This flow was first studied by T. Behrndt [7]. Instead of considering mean curvature flow in a Kähler–Einstein manifold, he considered the case when the ambient manifold is almost Einstein. Let us first recall the definition of an almost Einstein manifold in [7].

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Definition 1.1. An n-dimensional Kähler manifold $(M, J, \bar{\omega}, \bar{g})$ is called almost Einstein if

$$\bar{\rho} = \lambda\bar{\omega} + ndd^c\psi$$

for some constant $\lambda \in \mathbf{R}$ and some smooth function ψ on M.

Suppose the Kähler manifold $(M, J, \bar{\omega}, \bar{g})$ is almost Einstein. Given an immersion $F_0 : L \longrightarrow M$ of a manifold L into M, T. Behrndt [7] proposed the generalized mean curvature flow,

$$\begin{cases} \dfrac{\partial}{\partial t}F(x,t) = \mathbf{K}(x,t), \quad (x,t) \in L \times (0,T) \\ F(x,0) = F_0(x), \ x \in L. \end{cases} \tag{1.1}$$

Here

$$\mathbf{K} = \mathbf{H} - n\pi_{\nu L}(\bar{\nabla}\psi)$$

is a normal vector field along L which is called the generalized mean curvature vector field of L. As $\mathbf{K}$ is a differential operator differing from $\mathbf{H}$ just by lower order terms, it is easy to see that (1.1) has a unique solution on a short time interval [7].

Arguing in a similar way as Smoczyk did for Kähler–Einstein case [2], Behrndt [7] proved that if $L_0 = F_0(L)$ is Lagrangian in the almost Einstein manifold M, then along the generalized mean curvature flow (1.1), it remains Lagrangian for each time. Therefore, it is reasonable to call such a flow generalized Lagrangian mean curvature flow.

As a special case, Behrndt [7] also considered the generalized Lagrangian mean curvature flow in an almost Calabi–Yau manifold. Let us recall the definition of an almost Calabi–Yau manifold in [9].

Definition 1.2. An n-dimensional almost Calabi–Yau manifold $(M, J, \bar{\omega}, \bar{g}, \Omega)$ is an n-dimensional Kähler manifold $(M, J, \bar{\omega}, \bar{g})$ together with a non-vanishing holomorphic volume form Ω.

It can be seen that [7], there exists a smooth function ψ on an almost Calabi–Yau manifold M such that the Ricci form of $(M, \bar{g})$ is given by

$$\bar{\rho} = ndd^c\psi.$$

In particular, this implies that an almost Calabi–Yau manifold is almost Einstein.

Similar to the Calabi–Yau case, we can define the Lagrangian angle $\theta : L \to \mathbf{S}^1$ for a Lagrangian submanifold in an almost Calabi–Yau manifold, which satisfies [10]

$$F^*\Omega = e^{i\theta + n\psi}d\mu_g$$

for $F : L \to M$ a Lagrangian immersion. Note that θ is a multi-valued function on L, which is well-defined up to an additive constant $2k\pi$, $k \in \mathbf{Z}$. Behrndt [7] proved that on a Lagrangian submanifold of an almost Calabi–Yau manifold, we have

$$\mathbf{K} = J\nabla\theta. \tag{1.2}$$

Furthermore, along the generalized Lagrangian mean curvature flow, the Lagrangian angle satisfies (Proposition 5 of [7])

$$\frac{\partial}{\partial t}\theta = \Delta\theta + nd\psi(\nabla\theta). \tag{1.3}$$

We call a Lagrangian submanifold *almost calibrated* if $\cos\theta > 0$. When the Lagrangian angle θ is a single valued function, the Lagrangian L is called *zero-Maslov*. By (1.3) one can easily show that zero-maslov condition is preserved under the generalized Lagrangian mean curvature flow (1.1). It is obvious that almost calibrated Lagrangian must be zero-Maslov class. We call a Lagrangian submanifold L *Special Lagrangian* if $\theta \equiv \theta_0$ is a constant function on L (see Definition 5 and Proposition 3 of [7]). In this case, L is calibrated with respect to $Re(e^{-i\theta_0}\Omega)$ for the metric $\tilde{g} \equiv e^{2\psi}\bar{g}$.

Recall that L is a Lagrangian submanifold if $\bar{\omega}|_L \equiv 0$. Likewise, as in [4], we define an integral n-manifold L_1 and an integral n-current L_2 to be *Lagrangian* if

$$\int_{L_1} \phi|\omega \wedge \eta|d\mu = 0 \ \textit{for all } n-2 \textit{ form } \eta \textit{ and all smooth } \phi \in C^\infty(M)$$

and

$$\int_{L_2} \phi\omega \wedge \eta = 0 \ \textit{for all } n-2 \textit{ form } \eta \textit{ and all smooth } \phi \in C^\infty(M)$$

respectively. The concept of being Special Lagrangian can be easily extended to the case when L is an integral current.

It is known that, the mean curvature flow will blow up as the maximal norm of the second fundamental form blows up. According to the blow up rate, Huisken [11] divided the singularities of mean curvature flow into two types: Type-I and Type-II. Generally, singularity of mean curvature flow is unavoidable. Smoczyk (Theorem 2.3.5 of [12]) first proved that there is no compact Type-I singularities with zero-maslov class. Later, Chen–Li [3] and Wang [6] independently proved that there is no