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A note on stability of nongeneric equilibria for an underwater vehicle

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ABSTRACT

We study the Lyapunov stability of a family of nongeneric equilibria with spin for underwater vehicles with noncoincident centers. The nongeneric equilibria belong to singular symplectic leaves that are not characterized as a preimage to a regular value of the Casimir functions. We find an invariant submanifold such that the nongeneric equilibria belong to a preimage of a regular value that involves sub-Casimir functions. We obtain results for nonlinear stability on this invariant submanifold.

1. Introduction

We present a mathematical setting in which we can study the Lyapunov stability of nongeneric equilibria of a conservative dynamical system. We apply this setting to the case of nongeneric equilibria with spin for underwater vehicles with noncoincident centers.

Let (M, g) be a finite dimensional Riemannian manifold and $X \in \mathfrak{X}(M)$ a smooth vector field with x_e an equilibrium point for the dynamics generated by the vector field X. Our interest in this paper is to give sufficient conditions for Lyapunov stability of the equilibrium point in some degenerate case that will be explained in the following. Typically, for degenerate cases we do not obtain stability with respect to all the variables of the manifold M, but for some cases we can obtain stability with respect to a part of the variables determined by an invariant submanifold.

From the geometry of the problem a set of constraint functions $F_1, \dots, F_k : M \to \mathbb{R}$ is known. In the Hamilton–Poisson case of underwater vehicles with noncoincident centers, the constraint functions are given by Casimir functions and sub-Casimir functions, see [1–4]. A nongeneric equilibrium point x_e is a non-regular point for the function $\mathbf{F} = (F_1, \dots, F_k)$, i.e. $\mathbf{grad}\, F_1(x_e), \dots, \mathbf{grad}\, F_k(x_e)$ are linear dependent vectors in $T_{x_e}(M)$. In the context of Hamilton–Poisson systems nongeneric equilibria are points that belong to singular symplectic leaves and their stability have been extensively

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studied in [1,4–9]. Nevertheless, in some cases one can find a submanifold $x_e \in \widetilde{M} \subset M$ and a subset $F_{i_1}, \ldots, F_{i_q}$ of the constraint functions such that x_e is a regular point for $\widetilde{F}_{i_1} := F_{i_1|\widetilde{M}}, \ldots, \widetilde{F}_{i_q} := F_{i_q|\widetilde{M}} : \widetilde{M} \to \mathbb{R}$.

We introduce the submanifold $\widetilde{S}_e \subset \widetilde{M}$ as the preimage of the regular value $(\widetilde{F}_{i_1}(x_e), \ldots, \widetilde{F}_{i_q}(x_e)) \in \mathbb{R}^q$. We work in the hypothesis that $\widetilde{S}_e$ is invariant under the dynamics generated by X. The direct method of Lyapunov for the induced dynamics on $\widetilde{S}_e$ becomes: suppose there exists a smooth function $\widetilde{G}_e : \widetilde{S}_e \to \mathbb{R}$ such that:

(i) $\dot{\widetilde{G}}_e := d\widetilde{G}_e(X_{|\widetilde{S}_e}) \leq 0$,
(ii) $d\widetilde{G}_e(x_e) = 0$,
(iii) the Hessian matrix $\mathcal{H}^{\widetilde{G}_e}(x_e)$ is positive definite,

then x_e is a stable equilibrium point for the induced dynamics on the leaf $\widetilde{S}_e$.

A special case is when $\widetilde{M}$ is an invariant submanifold under the dynamics generated by the vector field X, $\widetilde{F}_{i_1}, \ldots, \widetilde{F}_{i_q}$ are conserved quantities for this induced dynamics, and $\widetilde{G} : \widetilde{M} \to \mathbb{R}$ is also a conserved quantity with the property that $\widetilde{G}_{|\widetilde{S}_e} = \widetilde{G}_e$. If moreover:

(a) $d(\widetilde{G}_{|\widetilde{S}_e})(x_e) = 0$,
(b) the Hessian matrix $\mathcal{H}^{\widetilde{G}_{|\widetilde{S}_e}}(x_e)$ is positive definite,

then x_e is a stable equilibrium point for the induced dynamics on the leaf $\widetilde{S}_e$.

The passage from stability for dynamics induced on an invariant regular leaf $\widetilde{S}_e$ to the ambient space $\widetilde{M}$ is a consequence of Arnold method [10], energy-Casimir method [11], Ortega–Ratiu method [12], algebraic method [13–15].

The condition (a) is equivalent with the following equality, see [16,17]:

(a′) $\mathbf{grad}\, \widetilde{G}(x_e) = \sum_{s=1}^{q} \sigma_{i_s}(x_e) \mathbf{grad}\, \widetilde{F}_{i_s}(x_e)$,

where $\sigma_{i_1}(x_e), \ldots, \sigma_{i_q}(x_e)$ are the Lagrange multipliers given by the formula

$$\sigma_{i_s}(x_e) := \frac{\det \Sigma_{(\widetilde{F}_{i_1}, \ldots, \widetilde{F}_{i_{s-1}}, \widetilde{G}, \widetilde{F}_{i_{s+1}}, \ldots, \widetilde{F}_{i_q})}^{(\widetilde{F}_{i_1}, \ldots, \widetilde{F}_{i_s}, \ldots, \widetilde{F}_{i_q})}(x_e)}{\det \Sigma_{(\widetilde{F}_{i_1}, \ldots, \widetilde{F}_{i_q})}^{(\widetilde{F}_{i_1}, \ldots, \widetilde{F}_{i_q})}(x_e)}, \tag{1.1}$$

with the Gramian matrix defined by

$$\Sigma_{(g_1, \ldots, g_s)}^{(f_1, \ldots, f_r)} = \begin{bmatrix} \langle \mathbf{grad}\, g_1, \mathbf{grad}\, f_1 \rangle & \ldots & \langle \mathbf{grad}\, g_s, \mathbf{grad}\, f_1 \rangle \\ \ldots & \ldots & \ldots \\ \langle \mathbf{grad}\, g_1, \mathbf{grad}\, f_r \rangle & \ldots & \langle \mathbf{grad}\, g_s, \mathbf{grad}\, f_r \rangle \end{bmatrix}. \tag{1.2}$$

The gradients in the above formulas are computed with respect to the induced metric on $\widetilde{M}$ by the ambient Riemannian space (M, g).

Further, the condition (b) is equivalent with the following equality, see [16]:

(b′) $\left[\mathcal{H}^{\widetilde{G}}(x_e)\right]_{|T_{x_e}\widetilde{S}_e \times T_{x_e}\widetilde{S}_e} - \sum_{s=1}^{q} \sigma_{i_s}(x_e) \left[\mathcal{H}^{\widetilde{F}_{i_s}}(x_e)\right]_{|T_{x_e}\widetilde{S}_e \times T_{x_e}\widetilde{S}_e}$ is positive definite,

where Hessian matrices are also computed with respect to the induced metric on $\widetilde{M}$.

Theorem 1.1. *Suppose that $\widetilde{M} \subset M$ is an invariant submanifold under the dynamics, $\widetilde{F}_{i_1}, \ldots, \widetilde{F}_{i_q}, \widetilde{G}$ are conserved quantities that satisfy conditions* (a′) *and* (b′) *then the nongeneric equilibrium point x_e is Lyapunov stable on the invariant space $\widetilde{M}$.*

In general, conditions (a′) and (b′) does not imply stability in the whole ambient space M.

2. Stability of nongeneric equilibria of underwater vehicles with noncoincident centers

Following [2], the dynamics of an underwater vehicle modeled as a neutrally buoyant, submerged rigid body in an infinitely large volume of irrotational, incompressible, inviscid fluid that is at rest at infinity is described by the system

$$\begin{cases} \dot{\boldsymbol{\Pi}} = \boldsymbol{\Pi} \times \boldsymbol{\Omega} + \mathbf{P} \times \mathbf{v} - mgl\boldsymbol{\Gamma} \times \mathbf{r} \\ \dot{\mathbf{P}} = \mathbf{P} \times \boldsymbol{\Omega} \\ \dot{\boldsymbol{\Gamma}} = \boldsymbol{\Gamma} \times \boldsymbol{\Omega}, \end{cases} \tag{2.1}$$

where $\boldsymbol{\Pi}$ is the angular impulse, $\mathbf{P}$ is the linear impulse, $\boldsymbol{\Gamma}$ is the direction of gravity, $l\mathbf{r}$ is the vector from center of buoyancy to center of gravity (with $l \geq 0$ and $\mathbf{r}$ a unit vector), m is the mass of the vehicle, g is gravitational acceleration, $\boldsymbol{\Omega}$ and $\mathbf{v}$