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Necessary and sufficient conditions for discrete wavelet frames in $\mathbb{C}^N$

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ABSTRACT

We present necessary and sufficient conditions with explicit frame bounds for a discrete wavelet system of the form $\{D_a T_k \phi\}_{a \in U(N), k \in I_N}$ to be a frame for the unitary space $\mathbb{C}^N$. It is shown that the canonical dual of a discrete wavelet frame for $\mathbb{C}^N$ has the same structure. This is not true (well known) for canonical dual of a wavelet frame for $L^2(\mathbb{R})$. Several numerical examples are given to illustrate the results.

1. Introduction

The purpose of this paper is to analyze the discrete wavelet structure of the form $\{D_a T_k \phi\}_{a \in U(N), k \in I_N}$ in $\mathbb{C}^N$, where D_a and T_k are dilation and translation operators on $\mathbb{C}^N$, respectively and $\phi \in \mathbb{C}^N$. Of course, there is an extensive literature on wavelet frames for $L^2(\mathbb{R}^d)$ and for some special types of function spaces and it is impossible to give complete references; let us at least mention some [1–7]. The main contributions of this paper are as follows: Firstly, we present a necessary condition for discrete wavelet frames for $\mathbb{C}^N$ in terms of a series associated with the Fourier transform of the window function, see Theorem 3.2. It is observed that the necessary condition given in Theorem 3.2 is also a sufficient condition for discrete wavelet frames in $\mathbb{C}$ and $\mathbb{C}^2$, see Proposition 3.3. Theorem 3.6 provides a sufficient condition for a family of vectors of the form $\{D_a T_k \phi\}_{a \in U(N), k \in I_N}$ to be a frame for $\mathbb{C}^N$. Chui and Shi proved in [8] that the canonical dual of a wavelet frame for $L^2(\mathbb{R})$ need not have a wavelet structure. The situation is different for discrete wavelet frames for $\mathbb{C}^N$. More precisely, the canonical dual of a discrete wavelet frame for $\mathbb{C}^N$ has the same structure, see Theorem 3.10.

Frames are redundant building blocks which provide a series representation (not necessarily unique) for each vector in the space. Duffin and Schaeffer [9] in 1952, introduced the concept of frame in the context of nonharmonic Fourier series. Throughout, $\mathbb{C}^N$ will denote an N-dimensional complex separable Hilbert space with inner product $\langle ., . \rangle$. A family of vectors $\mathcal{F} = \{\phi_k\}_{k=1}^M$ in $\mathbb{C}^N$ is called a *frame* (or *Hilbert frame*) for $\mathbb{C}^N$ if there exist constants $0 < a_o \le b_o < \infty$ such that

$$a_o \|x\|^2 \le \sum_{k=1}^{M} |\langle x, \phi_k \rangle|^2 \le b_o \|x\|^2 \text{ for all } x \in \mathbb{C}^N.$$

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The numbers a_o and b_o are called *lower* and *upper frame bounds*, respectively. If it is possible to choose $a_o = b_o$, then we say that $\mathcal{F}$ is *tight*. If $\mathcal{F}$ is a frame for $\mathbb{C}^N$, the frame operator $S : \mathbb{C}^N \to \mathbb{C}^N$ given by

$$Sx = \sum_{k=1}^{M} \langle x, \phi_k \rangle \phi_k, \ x \in \mathbb{C}^N$$

is a bounded, linear, positive and invertible operator on $\mathbb{C}^N$. Thus, each $x \in \mathbb{C}^N$ has the expansion

$$x = SS^{-1}x = \sum_{k=1}^{M} \langle S^{-1}x, \phi_k \rangle \phi_k = \sum_{k=1}^{M} \langle x, S^{-1}\phi_k \rangle \phi_k.$$

The scalars $\{\langle S^{-1}x, \phi_k \rangle\}$ are called *frame coefficients* of the vector $x \in \mathbb{C}^N$. The representation of f in the reconstruction formula need not be unique. Thus, frames allow each element in the space to be written as a linear combination of frame elements, where linear independence of frame elements is not required. Finite frames have potential applications in quantum mechanics [10,11]. Pfander studied Gabor frames on finite-dimensional complex vector spaces in [12]. Very recently, Deepshikha and Vashisht [13] discussed frame properties of a system of the form $\{T_k\phi\}_{k\in I_N}$ in $\mathbb{C}^N$. Thirulogasanthar and Bahsoun [14] discussed methods for constructing continuous, discrete and finite frames. They presented a method to obtain frames on fractals, by using a distance function. By using the iterated function systems (IFS), Thirulogasanthar and Bahsoun [14] obtained continuous and discrete frames, living on fractal sets, of both finite and infinite dimensional separable abstract Hilbert spaces. For more details about the link between frames and iterated function systems, we refer [15–17]. Discrete frames on a finite dimensional right quaternion Hilbert space were studied by Khokulan et al. in [18] (also see [19]). Application of frames in applied mathematics with different directions can be found in the books of Casazza and Kutyniok [10], Christensen [20,21], Daubechies [2], Han, Kornelson, Larson, and Weber [22] and Okoudjou [23].

2. Basic tools

We follow notations and definitions given in [12]. The symbol $\mathbb{C}$ will denote the set of complex numbers ; $\mathbb{Z}$ the set of all integers and N a positive integer. An arbitrary element x in the unitary space $\mathbb{C}^N$ is represented by $((x(0), x(1), \dots, x(N-1))^T$, where $x(n)$ is the $(n+1)$th component of the column vector x and x^T denotes the transpose of x.

That is

$$\mathbb{C}^N = \Big\{ (x(0), x(1), \dots, x(N-1))^T : x(i) \in \mathbb{C}, \ i \in I_N = \{0, 1, \dots, N-1\} \Big\}.$$

An element $p \in I_N$ is called a unit in I_N if it has a multiplicative inverse in I_N, that is, if there exists $q \in I_N$ such that $p.q = 1$, where multiplication is over modulo N. The set of units in I_N is denoted by $U(N)$. By $\Phi(N)$ we denote the number of units in I_N.

We consider the following linear operators on I_N. For $k \in I_N$, the *translation operator* $T_k : \mathbb{C}^N \to \mathbb{C}^N$ is given by

$$T_k(x(0), x(1), \dots, x(N-1))^T = (x(0-k), x(1-k), \dots, x((N-1)-k))^T,$$

where subtraction is over modulo N.

For $l \in I_N$, the *modulation operator* $M_l : \mathbb{C}^N \to \mathbb{C}^N$ is defined as

$$M_l(x(0), x(1), \dots, x(N-1))^T = (e^{2\pi i l 0/N}x(0), e^{2\pi i l 1/N}x(1), \dots, e^{2\pi i l(N-1)/N}x(N-1))^T.$$

Let $a \in U(N)$. The *dilation operator* $D_a : \mathbb{C}^N \to \mathbb{C}^N$ is given by

$$D_a(x(0), x(1), \dots, x(N-1))^T = (\, x(a.0), x(a.1), \dots, x(a.(N-1))\,)^T,$$

where multiplication is over modulo N.

The dilation operator D_a is a unitary operator. Indeed, for all $x, y \in \mathbb{C}^N$, we have

$$\langle D_a x, y \rangle = \sum_{n=0}^{N-1} x(a.n)\overline{y(n)} = \sum_{n=0}^{N-1} x(n)\overline{y(a^{-1}.n)} = \langle x, D_{a^{-1}}y \rangle.$$

Therefore, $D_a^* = D_{a^{-1}}$.

Furthermore

$$\begin{aligned} D_a^* D_a x &= D_a^*(\, x(a.0), x(a.1), \dots, x(a.(N-1))\,)^T \\ &= (\, x(a^{-1}.a.0), x(a^{-1}.a.1), \dots, x(a^{-1}.a.(N-1))\,)^T \\ &= (\, x(0), x(1), \dots, x(N-1)\,)^T \\ &= x \ \text{ for all } x \in \mathbb{C}^N. \end{aligned}$$

Hence the dilation operator D_a is unitary.