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Completeness of inextensible electromagnetic trajectories in a stationary spacetime

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ABSTRACT

A new technique is introduced to study the completeness of inextensible electromagnetic trajectories in an $n(\geq 2)$-dimensional stationary spacetime. Sufficient conditions on an electromagnetic field on a stationary spacetime are imposed to ensure that the associated $(n+1)$-dimensional Kaluza–Klein bundle spacetime is itself stationary. The problem is then reduced to the geodesic completeness of the corresponding Kaluza–Klein bundle spacetime. Applications are given specially to the case of a standard static spacetime.

1. Introduction

Let (M, g) be an $n(\geq 2)$-dimensional spacetime and F an electromagnetic field on M, i.e., a closed 2-form on the spacetime. Let us consider the following second order differential equation,

$$\frac{D\gamma'}{dt} = \frac{q}{m}\widetilde{F}(\gamma'), \tag{E}$$

where $\widetilde{F}$ the skew-adjoint (1, 1) tensor field defined by F via g, i.e., $F(X, Y) = g(X, \widetilde{F}(Y))$, for all $X, Y \in \mathfrak{X}(M)$, D/dt represents the Levi-Civita covariant derivative along γ, γ' the velocity of γ and $q \in \mathbb{R}$, $m > 0$ are constants.

Given $p \in M$ and an initial velocity $v \in T_pM$, there exists a unique (inextensible) solution $\gamma : I \longrightarrow M$, $0 \in I$, of Eq. (E) satisfying (see for instance [1, Th. 3.8.3]),

$$\gamma(0) = p, \qquad \gamma'(0) = v.$$

The differential equation (E), well-known in General Relativity, is called the Lorentz force equation (see for instance [1, Def. 3.8.1] and its solutions are called electromagnetic trajectories in spacetime. In the case $g(v, v) = -m^2$, from the skew-symmetry of F, any solution γ of Eq. (E) satisfies $g(\gamma'(t), \gamma'(t)) = -m^2$ everywhere and thus it represents a relativistic particle in spacetime [1, Def. 3.1.1]. Thus, the Lorentz force equation governs the dynamics of a relativistic

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charged particle, with mass $m > 0$ and electric charge q, in presence of an electromagnetic field F. Such particles will be called *electromagnetic trajectories* along the paper.

For a trivial electromagnetic field, i.e., $F = 0$, the solutions of Eq. (E) are indeed the geodesics of the spacetime (M, g). It is well-known that they are characterized as critical points of the energy functional (timelike geodesics represents free falling particles and lightlike geodesics photons in spacetime). However, if $F \neq 0$ then there exists no affine connection on M whose geodesics are the electromagnetic trajectories in spacetime [2, Prop. 2.1].

On the other hand, the electromagnetic field F is locally exact from the classical Poincaré Lemma. If F is assumed to be globally exact, i.e., $F = d\mu$ for some globally defined potential 1-form μ, then the same argument as in [2] shows that the Lorentz force equation is indeed the Euler–Lagrange equation of a natural variational problem.

In general, the interval of definition I of an inextensible solution γ of Eq. (E) is not the whole real line. When $I = \mathbb{R}$, the solution γ is said to be *complete*. In the case γ is a timelike curve, it may be interpreted as saying that the charged particle lives forever in spacetime. Our aim in this article is to find suitable assumptions on the electromagnetic field and on the spacetime in order to get that any inextensible electromagnetic trajectory is complete. Let us recall that the extendibility of the solutions to a certain second order differential equation on a Riemannian manifold, which formally extends Eq. (E) in this case, have been analyzed in [3]. The technique used in this paper strongly depends on the positive definite character of the metric and it is not directly generalizable to the Lorentzian case. It is proven in [4, Th. 1] that on a compact Lorentzian manifold (M, g) which admits a timelike conformal vector field K any inextensible solution of Eq. (E), for an electromagnetic field F such that the observers in the reference frame $K/\sqrt{-g(K,K)}$ perceive no electric vector field [1, p. 75], must be complete.

Our approach here is new and different from the one in [4]. In fact, we will reduce the completeness of the inextensible solutions of (E) on a Lorentzian manifold (M, g) to the geodesic completeness of another Lorentzian manifold $(\overline{M}, \overline{g})$, the Kaluza–Klein bundle spacetime associated to M and the corresponding electromagnetic field F, which is a $U(1)$-bundle and the total spacetime of a semi-Riemannian submersion on M. Next, after assuming the existence of a timelike infinitesimal isometry on M we look for reasonable assumptions on M to get the geodesic completeness mainly without assuming the compactness of M.

The content of the paper is organized as follows: Section 2 is devoted to recall several facts on Kaluza–Klein's theory. The infinitesimal symmetry imposed on the spacetime is introduced in Section 3, in particular the notion of standard static spacetime is recalled. In Section 4 the Kaluza–Klein semi-Riemannian submersion is studied and sufficient conditions to ensure that the Kaluza–Klein spacetime associated to a stationary spacetime with an electromagnetic field is itself stationary are obtained (Proposition 4.1). Finally, in Section 5 we prove our main and more general result on completeness of inextensible electromagnetic trajectories (Theorem 5.1). Applications are then derived to the case of a standard static spacetime (Corollary 5.2) and to the case, with only mathematical relevance, of a compact stationary spacetime (Corollary 5.5).

2. Preliminaries

A spacetime is a time orientable $n(\geq 2)$-dimensional Lorentzian manifold (M, g) endowed with one fixed time orientation. Along this paper we will denote a spacetime by M and, as usual, we will refer sometimes the points of M as events.

We recall that a *particle of mass* $m > 0$ in spacetime M is a (smooth) future pointing unit timelike curve $\gamma : I \longrightarrow M$, where I an open interval of the real line $\mathbb{R}$, which satisfies $g(\gamma'(t), \gamma'(t)) = -m^2$. A particle with $m = 1$ is called an *observer* and its parameter t represents then its proper time.

Let us consider an electrically charged particle, i.e., a triple (γ, m, q), where γ is a particle, $m > 0$ its mass and $q \in \mathbb{R}$ its charge, in presence of an electromagnetic field F, i.e., a closed 2-form on M. The dynamics of the particle is totally described by the Lorentz force equation (E), and the vector field $\widetilde{F}(\gamma')$ along γ is interpreted as the *electric field* relative to γ, [1, p. 75].

Electromagnetism may be described through a $U(1)$-principal fiber bundle over the spacetime M, $\pi : \overline{M} \longrightarrow M$. The electromagnetic field is then codified by means of a connection $\omega : T\overline{M} \longrightarrow \mathfrak{g}$, where $\mathfrak{g}(\cong\mathbb{R})$ is the Lie algebra of $U(1)$. In this context, the electromagnetic field on M essentially corresponds with the curvature form $\Omega := d\omega$ on $\overline{M}$, here d denotes the exterior covariant derivative, namely, $d\omega = (\pi^* F)\,\zeta$, where ζ is a fixed generator of $\mathfrak{g}$ (see for instance [5, 1.2.7]). This follows an old unifying idea due to Kaluza [6] and Klein [7] (see [5, Chap. 9] for a modern and general handling). It is assumed that the unobservable manifold $\overline{M}$ is endowed with a $U(1)$-invariant Lorentzian metric $\overline{g}$ of the form $\overline{g} = \pi^* g + \omega^* g_{\mathfrak{g}}$, that is,

$$\overline{g}(U, V) = g(d\pi(U), d\pi(V)) + g_{\mathfrak{g}}(\omega(U), \omega(V)), \tag{1}$$

where $U, V \in \mathfrak{X}(\overline{M})$ and $g_{\mathfrak{g}}$ is an adjoint-invariant positive definite inner product on $\mathfrak{g}$, normalized by $g_{\mathfrak{g}}(\zeta, \zeta) = 1$. Thus, from the Lorentzian manifold (M, g) and the electromagnetic field F we have a new Lorentzian manifold $(\overline{M}, \overline{g})$ such that the fundamental vector field D corresponding to ζ is Killing and $\overline{g}(D, D) = 1$, thus the vertical vector field D is spacelike. Therefore, the connection form may be then expressed as follows

$$\omega = \overline{g}(D, \cdot\,)\,\zeta. \tag{2}$$

Let us point out that the associated Kaluza–Klein bundle is unique [8], although the connection is not uniquely determined by F, and the map π is a semi-Riemannian submersion in the sense of [9].