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The Poincaré–Hopf Theorem for line fields revisited

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ABSTRACT

A Poincaré–Hopf Theorem for line fields with point singularities on orientable surfaces can be found in Hopf's 1956 Lecture Notes on Differential Geometry. In 1955 Markus presented such a theorem in all dimensions, but Markus' statement only holds in even dimensions $2k \geq 4$. In 1984 Jänich presented a Poincaré–Hopf theorem for line fields with more complicated singularities and focussed on the complexities arising in the generalized setting.

In this expository note we review the Poincaré–Hopf Theorem for line fields with point singularities, presenting a careful proof which is valid in all dimensions.

1. Introduction

A line field is a smooth assignment of a tangent line at each point of a manifold, and may be thought of as a projective analogue of a vector field. More generally a line field may have a singular set where it is undefined. Line fields have come to prominence recently in soft matter physics where they are known also as *nematic fields*, and their singularities as *topological defects*. In this setting they may be used to mathematically model certain types of ordered media. For example, nematic liquid crystals, which are materials formed of rod shaped molecules with no head or tail, can be reasonably modelled in this way [1,2]. Much of the topological interest in line fields lies in the study and classification of their singularities, and for this the tools of homotopy theory have proven to be useful. We recommend Mermin's influential essay [3] or the colloquium [4] of Alexander et al. as readable introductions to these ideas.

The classical Poincaré–Hopf theorem [5,6] relates the singularities of a vector field to the Euler characteristic of the underlying manifold. It states that for a vector field with finitely many isolated zeros on a compact manifold M, the sum of the indices at the zeros equals the Euler characteristic of M. There is an analogous but less well-known result for line fields with singularities, which is often quoted in the soft matter physics literature, and appears in the mathematical literature in various forms in works of Hopf, Markus, Koschorke and Jänich (see the discussion below).

In this article we give a careful proof of the following Poincaré–Hopf theorem for line fields with singularities.

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Theorem 1.1. *Let M^m be a compact manifold of dimension $m \geq 2$, and let ξ be a line field on M with finitely many singularities $x_1, \dots, x_n$. If $\partial M \neq \varnothing$, we assume additionally that the singularities lie in the interior of M, and that the line field is normal to ∂M. The projective index $\mathrm{p\,ind}_\xi(x_i)$ of each singularity is defined (see Definition 3.5); it is an integer if m is even, and an integer mod 2 if m is odd.*

Let $\chi(M)$ denote the Euler characteristic of M. We have

$$\sum_{i=1}^{n} \mathrm{p\,ind}_\xi(x_i) = 2\chi(M),$$

where the equality is interpreted as congruence mod 2 when m is odd.

Several statements similar to Theorem 1.1 can be found in the mathematical literature, dating back to at least the 1950s. Perhaps the first such appears in the lecture notes of Heinz Hopf [7, p. 113] (where Poincaré is credited), and is stated only for orientable surfaces. Another, due to Lawrence Markus, appeared in an article in the Annals of Mathematics [8, Theorem 2]. Although it is stated for all dimensions, counter-examples may be given for surfaces and odd-dimensional manifolds (see Examples 2.9 and 2.10).

Our contribution is to give a unified proof of Theorem 1.1 valid in all dimensions, thereby correcting the statement of [8, Theorem 2], and generalizing to higher dimensions the result in [7]. The proof we offer in Section 4 is a correction of the proof in [8]. One passes to a double branched cover associated to the line field which supports a vector field with isolated zeros, then applies the classical Poincaré–Hopf theorem and the Riemann–Hurwitz formula. The mistake made by Markus [8] in the surface case (and rediscovered by the present authors) was in identifying the vector field indices in the double cover in terms of the projective indices in the original manifold. We introduce normal indices in Section 3 in an attempt to clarify this rather subtle point.

In addition to the work of Hopf and Markus, Koschorke [9] and Jänich [10,11] have investigated line fields with singularities in great detail. Koschorke's results [9, Propositions 1.3 & 1.8] give a Poincaré-Hopf Theorem for line fields which implies Theorem 1.1 when $m > 2$, but Koschorke's definition of a singular line field is not the same as the one we use. He considers a line field to be a vector bundle morphism $v : \xi \to TM$ from a line bundle ξ on M to the tangent bundle TM, and its singularities to be the points where v drops rank. With this definition, every isolated singularity on a surface is orientable; i.e. has even projective index as defined in Definition 3.5. Consequently the difficulties arising in the surface case when a line field cannot be extended over a singularity do not arise in Koschorke's setting.

Jänich [10,11] investigates line fields with singularities from the viewpoint of obstruction theory (as suggested in [9, Remark 1.9]). His definition of a line field with singularities is more general than ours, in that he also considers the case where the singular set may have components of codimension two. Sections 1 and 2 of [11] contain a proof of Theorem 1.1 along the lines of Koschorke [9], but treating $m = 2$ as a special case. Jänich shows that in the surface case, the sum of the projective indices may be viewed as the Poincaré dual of the cohomology class obstructing the existence of a line field without singularities [11, Satz und Definition 1.3]. Hence this sum is independent of the particular line field. The value of the sum is then calculated by taking a line field which comes from a vector field [11, 2.3].

It is our hope that this paper generates interest in questions of algebraic and differential topology arising in the theory of soft matter physics.

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2. Definitions and previous results

Let M^m be a smooth manifold of dimension $m \geq 2$ and let $TM \to M$ be the tangent bundle of M. A vector field on M is a smooth section $v : M \to TM$. If a zero x of v is isolated one can define an integer $\mathrm{ind}_v(x_i)$, the *index* of v at x; see Definition 3.1. Recall that the Euler characteristic of a compact manifold M is defined to be the alternating sum of its Betti numbers:

$$\chi(M) := \sum_{i=0}^{\infty} (-1)^i \mathrm{rank}\left(H_i(M; \mathbb{Q})\right).$$

Let v be a vector field on the compact manifold M with finitely many zeros $\{x_1, \dots, x_n\} \subset M$. If M has a boundary, then we require v to be pointing outwards at all boundary points. The Poincaré-Hopf Theorem [5,6] states that the Euler characteristic of M agrees with the sum of the indices of v:

$$\chi(M) = \sum_{j=1}^{n} \mathrm{ind}_v(x_j).$$

The following related statement is well-known, and is also called the Poincaré-Hopf Theorem by some authors.

Proposition 2.1 (*[12, p. 552]*)**.** *A closed manifold M admits a non-vanishing vector field if and only if $\chi(M) = 0$.*