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The Dirac operator on manifold admitting parallel one-form

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ABSTRACT

In this note, we get estimates on the eigenvalues of the Dirac operator on the manifold admitting a nontrivial parallel one-form, in terms of the eigenvalues of the Laplace–Beltrami operator and the scalar curvature. These estimates are sharp, in the sense that, for the first eigenvalue, they reduce to the result of Alexandrov et al. (1998).

1. Introduction

We suppose that (M^n, g) is a closed Riemannian manifold with a fixed spin structure. We understand the spin structure as a reduction $SpinM^n$ of the SO(n)-principal bundle of M^n to the universal covering $Ad : Spin(n) \to SO(n)$ of the special orthogonal group. The spinor bundle $\Sigma M^n = SpinM^n \times_\rho \Sigma_n$ on M^n is the associated complex $2^{[\frac{n}{2}]}$ dimensional complex vector bundle, where ρ is the complex spinor representation. The tangent bundle TM^n can be regarded as $TM^n = SpinM^n \times_{Ad} \mathbb{R}^n$. Consequently, the Clifford multiplication on ΣM^n is the fibrewise action given by

$$\begin{aligned} \mu : TM^n \otimes \Sigma M^n &\longrightarrow \Sigma M^n \\ X \otimes \psi &\longmapsto X \cdot \psi. \end{aligned}$$

On the spinor bundle ΣM^n, one has a natural Hermitian metric, denoted as the Riemannian metric by $\langle \cdot, \cdot \rangle$. The spinorial connection on the spinor bundle induced by the Levi-Civita connection ∇ on M^n will also be denoted by ∇. The Hermitian metric $\langle \cdot, \cdot \rangle$ and spinorial connection ∇ are compatible with the Clifford multiplication μ. That is

$$\begin{aligned} X\langle \phi, \varphi \rangle &= \langle \nabla_X \phi, \varphi \rangle + \langle \phi, \nabla_X \varphi \rangle \\ \langle X \cdot \phi, X \cdot \varphi \rangle &= |X|^2 \langle \phi, \varphi \rangle \\ \nabla_X (Y \cdot \phi) &= \nabla_X Y \cdot \phi + Y \cdot \nabla_X \phi, \end{aligned}$$

for $\forall \phi, \varphi \in \Gamma(\Sigma M^n)$ and $\forall X, Y \in \Gamma(TM^n)$. Using a local orthonormal frame field $\{e_1, \ldots, e_n\}$, the spinorial connection ∇, the Dirac operator D and the twistor operator P, are locally expressed as

$$\nabla_{e_k} \psi = e_k(\psi) + \frac{1}{4} e_i \cdot \nabla_{e_k} e_i \cdot \psi \tag{1.1}$$

and

$$D\psi \triangleq e_i \cdot \nabla_{e_i} \psi, \tag{1.2}$$

$$P\psi \triangleq e_i \otimes \left(\nabla_{e_i} \psi + \frac{1}{n} e_i \cdot D\psi \right) \tag{1.3}$$

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which satisfy the following important relation

$$|\nabla\psi|^2 = |P\psi|^2 + \frac{1}{n}|D\psi|^2,$$

for any $\psi \in \Gamma(\Sigma M^n)$. (Throughout this paper, the Einstein summation notation is always adopted.)

Let $R_{X,Y}Z \triangleq (\nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]})Z$ be the Riemannian curvature of (M^n, g) and denote by $\mathcal{R}_{X,Y}\psi \triangleq (\nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]})\psi$ the spin curvature in the spinor bundle ΣM^n. They are related via the formula

$$\mathcal{R}_{X,Y}\psi = \frac{1}{4}g(R_{X,Y}e_i, e_j)e_i \cdot e_j \cdot \psi. \tag{1.4}$$

We also use the notation

$$R_{ijkl} \triangleq g(R_{e_i,e_j}e_k, e_l)$$

and $R_{ij} = \langle Ric(e_i), e_j\rangle \triangleq R_{ikkj}$, $Scal = R_{ii}$. With the help of the Bianchi identity, (1.4) implies

$$e_i \cdot \mathcal{R}_{e_j,e_i}\psi = -\frac{1}{2}Ric(e_j) \cdot \psi, \tag{1.5}$$

which in turn gives $2e_i \cdot e_j \cdot \mathcal{R}_{e_i,e_j}\psi = Scal\,\psi$. Hence one derives the well-known Schrödinger–Lichnerowicz formula

$$D^2 = \nabla^*\nabla + \frac{1}{4}ScalId, \tag{1.6}$$

where ∇^* is the formal adjoint of ∇ with respect to the natural Hermitian inner product on ΣM^n. The formula shows the close relation between $Scal$ and the Dirac operator D.

From (1.6), it follows easily that if λ is an eigenvalue of D, then

$$\lambda^2 \geqslant \frac{1}{4}Scal_{\min},$$

where $Scal_{\min} \triangleq \min_M Scal$. Clearly, this inequality is interesting only for manifolds with positive scalar curvature, but the minimal value $\frac{1}{4}Scal_{\min}$ cannot be achieved for such manifolds.

The problem of finding optimal lower bounds for the eigenvalues of the Dirac operator on closed manifolds was for the first time considered in 1980 by Friedrich. Using the Lichnerowicz formula and a modified spin connection, he proved the following sharp inequality:

$$\lambda^2 \geqslant c_n Scal_{\min}, \tag{1.7}$$

where $c_n = \frac{n}{4(n-1)}$. The case of equality in (1.7) occurs iff (M^n, g) admits a nontrivial spinor field ψ called a real *Killing spinor*, satisfying the following overdetermined elliptic equation

$$\nabla_X\psi = -\frac{\lambda}{n}X \cdot \psi, \tag{1.8}$$

where $X \in \Gamma(TM)$ and the dot "." indicates the Clifford multiplication. The manifold must be a locally irreducible Einstein manifold.

The dimension dependent coefficient $c_n = \frac{n}{4(n-1)}$ in the estimate can be improved if one imposes geometric assumptions on the metric. Kirchberg showed that for Kähler metrics c_n can be replaced by $\frac{n+2}{4n}$ if the complex dimension $\frac{n}{2}$ is odd, and by $\frac{n}{4(n-2)}$ if $\frac{n}{2}$ is even. Alexandrov, Grantcharov, and Ivanov [1] showed that if there exists a parallel one form on M^n, then c_n can be replaced by c_{n-1}. Later, Moroianu and Ornea [2] weakened the assumption on the 1-form from parallel to harmonic with constant length. Note the condition that the norm of the 1-form being constant is essential, in the sense that the topological constraint alone (the existence of a non-trivial harmonic 1-form) does not allow any improvement of Friedrich's inequality (see [3]).

Another natural way to study the Dirac eigenvalues consists in comparing them with those of other geometric operators. Hijazi's inequality is already of that kind. As for spectral comparison results between the Dirac and the scalar Laplace operators, the first ones were proved by M. Bordoni. They rely on a very nice general comparison principle between two operators satisfying some kind of Kato-type inequality. Bordoni's results were generalized by Bordoni and Hijazi in the Kähler setting [4].

In this note, we find a new method which can recover the result of [1] and [2], moreover enable us to use the general spectral result of Bordoni, to show the following theorem.

Theorem 1. *Suppose there exists a non-trivial parallel one form on an n-dimensional closed Riemannian spin manifold* (M^n, g), $n \geq 3$ *and* $Scal \geq 0$. *Let* $\lambda_\alpha := \lambda_\alpha(D)$ *and consider the first N nonnegative eigenvalues,* $0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_N$, *then*