Journal of Geometry and Physics ▮ (▮▮▮▮) ▮▮▮–▮▮▮

Contents lists available at ScienceDirect

Journal of Geometry and Physics

journal homepage: www.elsevier.com/locate/jgp

Gauge theories on compact toric surfaces, conformal field theories and equivariant Donaldson invariants

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ARTICLE INFO

Article history:
Received 23 June 2016
Received in revised form 12 December 2016
Accepted 9 January 2017
Available online xxxx

Keywords:
Exact partition function
Supersymmetry
Equivariant localization
Donaldson invariants
Virasoro conformal blocks
AGT

ABSTRACT

We show that equivariant Donaldson polynomials of compact toric surfaces can be calculated as residues of suitable combinations of Virasoro conformal blocks, by building on AGT correspondence between $\mathcal{N} = 2$ supersymmetric gauge theories and two-dimensional conformal field theory.

Talk[1] presented by A.T. at the conference *Interactions between Geometry and Physics — in honor of Ugo Bruzzo's 60th birthday* 17–22 August 2015, Guarujá, São Paulo, Brazil, mostly based on Bawane et al. (0000) and Bershtein et al. (0000).

1. Introduction

A link between $\mathcal{N} = 2$ supersymmetric gauge theories in four dimensions and Donaldson invariants, classifying differentiable structures on four-manifolds, has been established since the seminal paper by Witten [1]. Here we exploit recent progresses both in the formulation of supersymmetric quantum field theories on curved spaces and in equivariant localization applied to supersymmetric path integrals in order to provide a direct computation of *equivariant* Donaldson polynomials for compact toric surfaces along the lines suggested by [2]. A crucial new ingredient along this path is provided by the correspondence between supersymmetric field theories in four dimensions and two-dimensional conformal field theories [3]. Indeed, our final result is that equivariant Donaldson polynomials on a compact toric manifold X can be expressed as residues of suitable combinations of Virasoro conformal blocks. This follows from the fact that the supersymmetric path integral reduces to a contour integral over a product of toric patches contributions given by Nekrasov partition functions [4]. These latter depend on the weights of the toric action through effective Omega-background parameters and correspond to Virasoro conformal blocks whose central charge depend on these parameters. The integration variable is the v.e.v. of the

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[1] http://salafrancesco.altervista.org/wugo2015/tanzini.pdf.

http://dx.doi.org/10.1016/j.geomphys.2017.01.012
0393-0440/

Please cite this article in press as: M. Bershtein, et al., Gauge theories on compact toric surfaces, conformal field theories and equivariant Donaldson invariants, Journal of Geometry and Physics (2017), http://dx.doi.org/10.1016/j.geomphys.2017.01.012

scalar field of the $\mathcal{N}=2$, $d=4$ vector multiplet, which on compact manifolds is a normalizable mode to be integrated over in the space of admissible classical BPS solutions. The latter are parameterized also by magnetic fluxes labeled by an integral lattice. A subtle issue concerns the allowed values of these fluxes, reflecting the stability conditions to be imposed on the equivariant vector bundles. We fully solved this problem for $X=\mathbb{P}^2$ in [5] and we present here some recent progress on $X=\mathbb{P}^1\times\mathbb{P}^1$.

We also consider maximally supersymmetric gauge theories and express the resulting partition functions, generating the Euler characteristics of the moduli spaces of stable equivariant vector bundles, in terms of mock modular forms. It is known from [6] that these partition functions, labeled by the first Chern class of the bundle, form a non-trivial representation of the modular group $SL(2,\mathbb{Z})$, obeying an analogue of the Verlinde algebra satisfied by the conformal blocks of rational conformal field theories in two dimensions.

On the other hand, it is known that knot invariants can be calculated by suitable gluing of Wess–Zumino–Witten conformal blocks [7]. The result we find here is somewhat analogous in the fact that equivariant Donaldson invariants are obtained from the gluing of conformal blocks of a *non-rational* conformal field theory. It would be interesting to further investigate this analogy for example by considering the insertion of surface operators in the supersymmetric path integral, which are known to be related to $\widehat{sl}(2)$ conformal blocks [8]. Surface operators in four-dimensions provide indeed co-boundary operators for knots [9].

2. Supersymmetry on compact manifolds

The symmetry group for the $\mathcal{N}=2$ supersymmetric theory on $\mathbb{R}^4$ is given by the rotation group $SO(4)=SU(2)_1\times SU(2)_2$ and the R-symmetry group $SU(2)_R\times U(1)_R$.

The supersymmetry is generated by the operator $\mathcal{Q}=\xi^{A\alpha}Q_{A\alpha}+\bar{\xi}^{A}_{\dot\alpha}\bar{Q}^{\dot\alpha}_{A}$ where the generators $\xi,\bar\xi$ are commuting Weyl spinors of $SU(2)_{1,2}$ (indices $\alpha,\dot\alpha$) respectively and moreover both of them transform in the two-dimensional representation of $SU(2)_R$ (index A).

On a generic manifold the covariantly constant condition $D\xi=0$ is too restrictive.

The consistency of the $\mathcal{N}=2$ supersymmetry algebra on a compact four manifold requires [10] that the spinor parameters have to satisfy the generalized Killing equations

$$\begin{aligned} &D_\mu\xi_B+T^{\rho\sigma}\sigma_{\rho\sigma}\sigma_\mu\bar\xi_B-\frac{1}{4}\sigma_\mu\bar\sigma_\nu D^\nu\xi_B=0\\ &D_\mu\bar\xi_B+\bar T^{\rho\sigma}\bar\sigma_{\rho\sigma}\bar\sigma_\mu\xi_B-\frac{1}{4}\bar\sigma_\mu\sigma_\nu D^\nu\bar\xi_B=0 \end{aligned}\tag{1}$$

and the auxiliary equations

$$\begin{aligned} \sigma^\mu\bar\sigma^\nu D_\mu D_\nu\xi_A+4D_\lambda T_{\mu\nu}\sigma^{\mu\nu}\sigma^\lambda\bar\xi_A&=M_1\xi_A,\\ \bar\sigma^\mu\sigma^\nu D_\mu D_\nu\bar\xi_A+4D_\lambda\bar T_{\mu\nu}\bar\sigma^{\mu\nu}\bar\sigma^\lambda\xi_A&=M_2\bar\xi_A, \end{aligned}\tag{2}$$

where background fields appear: two scalar M_1, M_2 a self-dual tensor $T_{\mu\nu}$, an anti-self-dual tensor $\bar T_{\mu\nu}$ and the connection of the $SU(2)_R$ R-symmetry bundle hidden in the covariant derivatives D_μ.

This is in agreement with the result of [11,12] obtained from supergravity in the spirit of [13,14].

2.1. Four-manifolds admitting a $U(1)$-isometry and the equivariant topological twist

For a general four-manifold it is always possible to solve the generalized Killing equations at least for one scalar supercharge ($\bar\xi^{\dot\alpha}_A=\delta^{\dot\alpha}_A$, $\xi_{A\alpha}=0$), performing a topological twist [1]. The spinors parameters are sections of the bundles

$$\xi\in\Gamma\left(S^+\otimes\mathcal{R}\otimes\mathcal{L}_R\right)\qquad\bar\xi\in\Gamma\left(S^-\otimes\mathcal{R}^\dagger\otimes\mathcal{L}_R^{-1}\right)\tag{3}$$

where $S^\pm$ are the spinor bundles of chirality $\pm$, $\mathcal{R}$ is the $SU(2)$ R-symmetry vector bundle and $\mathcal{L}_R$ is the $U(1)$ R-symmetry line bundle. Choosing $\mathcal{L}_R=\mathcal{O}$ to be the trivial line bundle and $\mathcal{R}=S^-$ realizes the topologically twisted theory, that is matching the R-symmetry connection with the spin connection of the manifold. For this choice of the R-symmetry bundles, $S^+\otimes S^-\sim T$ and $S^-\otimes S^-\sim\mathcal{O}+T^{(2,+)}$ with T the tangent bundle and $T^{(2,+)}$ the bundle of self-dual two-forms.

If the manifold has a $U(1)$-isometry generated by the vector field V a more general solution is available [4,12,10].

$$\bar\xi^{\dot\alpha}_A=\delta^{\dot\alpha}_A,\qquad\xi_{A\alpha}=V^\mu(\sigma_\mu)_{\alpha\dot\alpha}\xi^{\dot\alpha}_A,\qquad T=-\frac{1}{32}(d\zeta)^-,\qquad\bar T=M_1=M_2=0,\tag{4}$$

where ζ is a $U(1)$-invariant one-form such that $\iota_V\zeta=(V,V)$.

The supercharge generated by this solution has a scalar Q and a vector-like Q_μ component

$$\mathcal{Q}=Q+V^\mu Q_\mu\tag{5}$$

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