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Differential geometry of moduli spaces of quiver bundles

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ABSTRACT

Let P be a parabolic subgroup of a semisimple affine algebraic group G defined over $\mathbb{C}$ and X a compact Kähler manifold. L. Álvarez-Cónsul and O. García-Prada associated to these a quiver Q and representations of Q into holomorphic vector bundles on X (Álvarez-Cónsul and García-Prada, 2003) [1, 2]. Our aim here is to investigate the differential geometric properties of the moduli spaces of representations of Q into vector bundles on X. In particular, we construct a Hermitian form on these moduli spaces. A fiber integral formula is proved for this Hermitian form; this fiber integral formula implies that the Hermitian form is Kähler. We compute the curvature of this Kähler form. Under an assumption which says that X is a complex projective manifold, this Kähler form is realized as the curvature of a certain determinant line bundle equipped with a Quillen metric.

1. Introduction

Fix a semisimple affine algebraic group G defined over $\mathbb{C}$ and a proper parabolic subgroup $P \subset G$. L. Álvarez-Cónsul and O. García-Prada constructed a quiver Q with relations and established an equivalence of categories between the following two:

(1) holomorphic finite dimensional representations of P;
(2) finite dimensional representations of Q

(see [1]). Let X be a compact connected Kähler manifold equipped with a Kähler form. In [1], the authors proved the following more general result: There is an equivalence of categories between the following two:

(1) G–equivariant holomorphic vector bundles on $X \times (G/P)$;
(2) representations of Q into holomorphic vector bundles on X.

The representations of Q into holomorphic vector bundles on X are called quiver bundles on X. The notion of stability of vector bundles on X extends to quiver bundles on X. It is known that polystable quiver bundles carry a generalization of the Hermite–Einstein metrics [1,2] that are solutions of certain equations of *vortex type*.

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Given a holomorphic family of stable quiver bundles on X parameterized by a complex analytic space T, we construct a Hermitian structure on T; see Definition 3.1. This construction entails building the deformations of quiver bundles (this is carried out in Section 3.2) and relating the curvature of a generalized Hermite–Einstein metric to deformations (this is carried out in Section 3.3).

One of the main results here is a fiber integral formula for the above mentioned Hermitian structure on T; see Propositions 4.1 and 4.2. As a corollary we obtain that the Hermitian structure is Kähler (Corollary 4.1).

Another main result is a computation of this Kähler form on the moduli space of stable quiver bundles; this is carried out in Theorem 6.1.

Now assume that X is a complex projective manifold and the Kähler form on it is integral. We construct a holomorphic Hermitian line bundle on the moduli space of stable quiver bundles with the property that the corresponding Chern form coincides with the Kähler form on the moduli space; see Theorem 5.2. The construction of this holomorphic Hermitian line bundle is modeled on the works Quillen, [3], and Bismut–Gillet–Soulé [4].

2. Quiver bundles—notation and fundamental properties

2.1. Representations of quivers

We follow the notation of [1]. Two sets Q_0 and Q_1 together with two maps $h, t : Q_1 \longrightarrow Q_0$ give rise to a *directed graph* or *quiver*. The elements of Q_0 are called vertices, and the elements of Q_1 are called arrows. For $a \in Q_1$ the vertices $ha := h(a)$ and $ta := t(a)$ are called head of a and tail of a respectively. For an arrow a there is the notation $a : v \to v'$, where $v = ta$ and $v' = ha$. At this point the set of vertices Q_0 is not assumed to be finite. However the quiver $Q := (Q_0, Q_1)$ will have to be *locally finite* meaning for any vertex $v \in Q_0$ the sets $h^{-1}(v)$ and $t^{-1}(v)$ are assumed to be finite. A (non-trivial) path in Q is a sequence $p = a_0 \cdots a_m$ of arrows $a_j \in Q_1$ such that $ta_{i-1} = ha_i$ for $i = 1, \ldots, m$:

$$p : \overset{tp}{\bullet} \xrightarrow{a_m} \bullet \xrightarrow{a_{m-1}} \cdots \xrightarrow{a_0} \overset{hp}{\bullet} . \tag{2.1}$$

The vertices $tp = ta_m$ and $hp = ha_0$ are respectively called tail and head of the path p. By definition, the trivial path e_v at $v \in Q_0$ is equal to $e_v : v \to v$ in the above alternative notation.

A formal, finite sum

$$r = c_1p_1 + \cdots + c_\ell p_\ell$$

of paths p_j with complex coefficients is called a *(complex) relation* of a quiver. A *quiver with relations* is a pair $(Q, \mathcal{K})$, where Q is a quiver with $\mathcal{K}$ being a set of relations.

A *linear representation* $\mathbf{R} = (\mathbf{V}, \boldsymbol{\varphi})$ of a quiver Q is given by a collection $\mathbf{V}$ of complex vector spaces V_v for all vertices $v \in Q_0$ together with a collection $\boldsymbol{\varphi}$ of linear maps $\varphi_a : V_{ta} \to V_{ha}$ for all $a \in Q_1$. For all but finitely many vertices v the spaces V_v are required to be zero. Morphisms $\mathbf{f} : \mathbf{R} \to \mathbf{R}$ between representations $\mathbf{R} = (\mathbf{V}, \boldsymbol{\varphi})$ and $\mathbf{R}' = (\mathbf{V}', \boldsymbol{\varphi}')$ by definition consist of linear maps $f_v : V_v \to V'_v$ for all $v \in Q_0$ such that $\varphi_a \circ f_{ta} = f_{ha} \circ \varphi_a$. Given a representation $\mathbf{R} = (\mathbf{V}, \boldsymbol{\varphi})$, any non-trivial path p in the sense of (2.1) induces a linear map

$$\varphi(p) := \varphi_{a_0} \circ \cdots \circ \varphi_{a_m} : V_{tp} \to V_{hp}.$$

The linear map $\varphi(e_v)$ that is induced by the trivial path at a vertex v is by definition the identity $id : V_v \to V_v$.

A linear representation $\mathbf{R} = (\mathbf{V}, \boldsymbol{\varphi})$ is said to satisfy a relation $r = c_1p_1 + \cdots c_\ell p_\ell$, if

$$c_1\varphi(p_1) + \cdots + c_\ell\varphi(p_\ell) = 0.$$

For any set $\mathcal{K}$ of relations of a quiver Q a $(Q, \mathcal{K})$*-module* is a linear representation of Q that satisfies all relations from $\mathcal{K}$.

2.2. P-modules and quiver representations

Let G be a connected semisimple affine algebraic group defined over the complex numbers. Let P be a proper parabolic subgroup of G. In [1, p. 8, Section 1.3], a quiver Q with relations $\mathcal{K}$ is constructed from P. We will adopt the notation of [1, Section 1.3]. The reader is referred to [1, Section 1.3] for the details.

In [1, pp. 8–9, Theorem 1.4] the following equivalence of categories is proved:

$$\left\{ \begin{matrix} \text{finite dimensional holomorphic} \\ \text{representations of } P \end{matrix} \right\} \longleftrightarrow \left\{ \begin{matrix} \text{finite dimensional representations} \\ \text{of } Q \text{ satisfying relations in } \mathcal{K} \end{matrix} \right\}.$$