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Diophantine equations, Platonic solids, McKay correspondence, equivelar maps and Vogel's universality

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ABSTRACT

We notice that one of the Diophantine equations, $knm = 2kn + 2km + 2nm$, arising in the universality originated Diophantine classification of simple Lie algebras, has interesting interpretations for two different sets of signs of variables. In both cases it describes "regular polyhedra" with k edges in each vertex, n edges of each face, with total number of edges $|m|$, and Euler characteristics $\chi = \pm 2$. In the case of negative m this equation corresponds to $\chi = 2$ and describes true regular polyhedra, Platonic solids. The case with positive m corresponds to Euler characteristic $\chi = -2$ and describes the so called equivelar maps (charts) on the surface of genus 2. In the former case there are two routes from Platonic solids to simple Lie algebras—abovementioned Diophantine classification and McKay correspondence. We compare them for all solutions of this type, and find coincidence in the case of icosahedron (dodecahedron), corresponding to E_8 algebra. In the case of positive k, n and m we obtain in this way the interpretation of (some of) the mysterious solutions (Y-objects), appearing in the Diophantine classification and having some similarities with simple Lie algebras.

1. Introduction

Since ancient time natural numbers have been suggested as a basic notion in the construction of our knowledge about nature. However, it is rare when they are the part of the basis of construction of a given mathematical or physical theory. In this paper we consider several such cases and observe that they are based on the same Diophantine equation. Moreover, since two of these cases are connected to simple Lie algebras, we are naturally led to a comparison of these two superficially disjoint theories.

The focus of the present paper is the following Diophantine equation

$$\frac{1}{k}+\frac{1}{n}+\frac{1}{m}=\frac{1}{2}, \quad k, n, m \in \mathbb{Z}\setminus 0 \tag{1}$$

or in more general form, which allows zero values of integers k, n, m:

$$knm = 2kn + 2km + 2nm. \tag{2}$$

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Table 1
McKay correspondence and Diophantine classification.

Solutions (k, n, m)	Platonic solids	Subgroups of $SU(2)$	McKay correspondence	Diophantine classification
$(5, 3, -30)$	Icosahedron			
$(3, 5, -30)$	Dodecahedron	$2I$, $\|2I\| = 120$	E_8	E_8
$(4, 3, -12)$	Cube			
$(3, 4, -12)$	Octahedron	$2O$, $\|2O\| = 48$	E_7	E_6
$(3, 3, -6)$	Tetrahedron	$2T$, $\|2T\| = 24$	E_6	$SO(8)$
$(2, n, -n)$	n-polygon	C_n, $\|C_n\| = n$	A_{n-1}	A_n
		C_{2n}, $\|C_{2n}\| = 2n$	A_{2n-1}	
		BD_{2n}, $\|BD_{2n}\| = 4n$	D_{n-2}	
$(0, 0, 0)$				$D_{2,1,\lambda}$

We would like to point out that this equation appears in three circumstances, depending particularly on the signs of integers k, n, m. In two of them there are (different) routes from this equation to simple Lie algebras.

One route is the famous McKay correspondence [1]. It is well-known that solutions of Eq. (1) with $(+ + -)$ signs of variables describe Platonic solids (see below in Section 2). Take the invariance subgroup of a given Platonic solid (it is finite subgroup of the group $SO(3)$). Lift it to the group $SU(2)$ by double-covering map

$$1 \to \mathbb{Z}_2 \to SU(2) \to SO(3) \to 1, \tag{3}$$

and assign to this subgroup of $SU(2)$ by McKay procedure the simple Lie algebra from the list of ADE algebras (see Section 3). Note that one has to consider also degenerate "Platonic solids", and take into account different liftings of groups. All that is briefly described in Section 3.

An other route from Diophantine equation (1), with the same set of signs, to simple Lie algebras is given by recently developed [2] Diophantine classification of simple Lie algebras, based on Vogel's universality [3,4] and Deligne's conjecture on exceptional simple Lie algebras [5]. This is briefly described in Section 4.

In Section 5 we compare these two routes from solutions of Diophantine equation (1) to simple Lie algebras, and find several common features and differences.

Finally, we discuss relation of Diophantine equations (1) with the theory of equivelar maps [6–8] on orientable surfaces of genus two. They appear to correspond to the same Eq. (1) with $(+ + +)$ signs of variables. In Diophantine classification this case corresponds to mysterious Y-objects, which have certain similarity with simple Lie algebras, but up to now were not identified with any known objects. This is discussed in Section 6.

2. Platonic solids' Diophantine equation

Consider Platonic solid with number of edges of any face r, number of edges at any vertex n, total number of edges E, total number of vertices V, and total number of faces F. We have

$$nV = 2E\,, \quad rF = 2E\,.$$

Then Euler's theorem

$$V - E + F = 2$$

can be rewritten as

$$\frac{1}{r} + \frac{1}{n} - \frac{1}{E} = \frac{1}{2}. \tag{4}$$

This is the particular case of Diophantine equation (1) with the special choice $(+ + -)$ of signs of integers k, n, m.

Solutions (r, n, E) of Eq. (4) are:

- $(5, 3, 30)$ or $(3, 5, 30)$—dodecahedron or icosahedron,
- $(4, 3, 12)$ or $(3, 4, 12)$—cube (hexahedron) or octahedron,
- $(3, 3, 6)$—tetrahedron,
- $(2, n, n)$ (or, the same, $(r, 2, r)$)—regular n-polygon.

This information is listed in the first and second column of Table 1.

3. McKay correspondence

McKay correspondence assigns to finite subgroups of $SU(2)$ group Dynkin diagrams of some simple Lie algebras in the following way. Let G be an arbitrary finite subgroup of the group $SU(2)$ and let V be the restriction of 2-dimensional