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# Rational sphere valued supercocycles in $M$-theory and type IIA string theory

Domenico Fiorenza [a], Hisham Sati [b,c,*], Urs Schreiber [d]

[a] Dipartimento di Matematica, La Sapienza Università di Roma Piazzale Aldo Moro 2, 00185 Rome, Italy

[b] University of Pittsburgh, Pittsburgh, PA 15260, USA

[c] New York University, Abu Dhabi, United Arab Emirates

[d] Mathematics Institute of the Academy, Žitna 25, 115 67 Praha 1, Czech Republic[1]

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ABSTRACT

We show that supercocycles on super $L_\infty$-algebras capture, at the rational level, the twisted cohomological charge structure of the fields of $M$-theory and of type IIA string theory. We show that rational 4-sphere-valued supercocycles for M-branes in $M$-theory descend to supercocycles in type IIA string theory with coefficients in the free loop space of the 4-sphere, to yield the Ramond–Ramond fields in the rational image of twisted $K$-theory, with the twist given by the $B$-field. In particular, we derive the M2/M5 $\leftrightarrow$ F1/D$p$/NS5 correspondence via dimensional reduction of sphere-valued super-$L_\infty$-cocycles.



## 1. Introduction

(Super)cocycles play an important role in the study of the geometric and topological structures associated with physical theories (see [1] for an earlier survey). In [2] we discussed cocycles of super $L_\infty$-algebras (super Lie $n$-algebras for arbitrary $n$) forming the *brane bouquet* that gives the WZW terms of all the Green–Schwarz sigma models for all the branes in string theory and $M$-theory. This includes those with gauge fields on their worldvolume, the D-branes and the M5-brane, which were missing in the classical *brane scan*.

In [3] we had shown that this approach allows deriving the rational image of a twisted cohomology theory that unifies the M2-brane charges and the M5-brane charges (this is recalled in Section 2). Rationally this cohomology theory turns out to be represented by the 4-sphere, hence is *cohomotopy* in degree 4. This is in higher analogy to the familiar statement that the unification of D$p$-brane charges with the F1-brane charge ought to be in twisted K-cohomology theory. (That the fields of $M$-theory should take values in the 4-sphere was first suggested in [4,5].)

In Section 3 we show, at the rational level, that indeed the twisted M2/M5 charges in degree-4 cohomotopy in 11 dimensions dimensionally reduce to the twisted $K$-theory of the F1/D$p$/NS5-brane charges in 10 dimensions (for $p \in \{0, 2, 4\}$), where the dimensionally reduced cohomology theory is represented by the rationalization of the homotopy quotient $\mathcal{L}S^4//S^1$ of the free loop space of the 4-sphere. In particular this exhibits a purely $L_\infty$-theoretic derivation, at the rational level, of twisted $K$-theory as the home of the brane charges in type II string theory. The lift of this twisted charge

* Corresponding author at: University of Pittsburgh, Pittsburgh, PA 15260, USA.
*E-mail address:* hsati@nyu.edu (H. Sati).

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structure to $M$-theory has been an open problem. This may be viewed as one confirmation at the rational level of the proposal in [4,5] on the description of $M$-theory via twisted generalized cohomology.

In the existing literature, the cocycles for the WZW terms of the D$p$-branes are instead constructed separately as independent cocycles on extended super-Minkowski spacetime (see [3] for references and for the super $L_\infty$-algebraic formulation). In Section 4 we show that the same $L_\infty$-descent mechanism which unifies the M2- and M5-brane charges also applies to the separate D$p$-brane cocycles, and they descend to again a single cocycle with coefficients in (the rational image of) the relevant truncation of twisted $K$-theory.

The techniques that we use are from geometric homotopy theory [6], cast in computationally powerful algebraic language. Lecture notes accompanying the discussion here may be found in [7]. We consider super $L_\infty$-algebras as in [8,2]. These are generalizations of super Lie algebras to super Lie $n$-algebras, for arbitrary $n$, where instead of just a super Lie bracket we have brackets of all arities with the Lie bracket being the binary one. More precisely, our construction takes place in the homotopy category of super $L_\infty$-algebras, so that a morphism from a super $L_\infty$-algebra $\mathfrak{g}$ to a super $L_\infty$-algebra $\mathfrak{h}$ will actually be a span of morphism

$$\mathfrak{g} \xleftarrow{\sim} \tilde{\mathfrak{g}} \to \mathfrak{h}$$

where $\tilde{\mathfrak{g}} \xrightarrow{\sim} \mathfrak{g}$ is a quasi-isomorphism, i.e., an $L_\infty$-morphism inducing an isomorphism of graded vector spaces at the level of cohomology from $H^\bullet(\tilde{\mathfrak{g}})$ to $H^\bullet(\mathfrak{g})$. Passing from $\mathfrak{g}$ to $\tilde{\mathfrak{g}}$ is an example of resolution. This concept has many incarnations, depending on the context (homotopic, fibrant, cofibrant, projective, injective). For us, what is important is that is a concept of equivalence within a category between the object at hand and another (or a combination of such) that generally behaves in a more utilizable way within the same category.

Furthermore, we will make constant use of the duality between (finite type) super $L_\infty$-algebras and differential graded-commutative super-algebras, identifying a super $L_\infty$-algebra $\mathfrak{g}$ with its Chevalley–Eilenberg algebra $\mathrm{CE}(\mathfrak{g})$ as in [8]. These Chevalley–Eilenberg algebras of super $L_\infty$-algebras are what are called *FDA*s in the supergravity literature (going back to [9]). The point of identifying these as dual to super $L_\infty$-algebras is to make manifest their higher gauge theoretic nature and the relevant homotopy theory, which is crucial for the results we present here. For instance, for $p \in \mathbb{N}$, the line $(p+2)$-algebra $b^{p+1}\mathbb{R}$, i.e., the chain complex with $\mathbb{R}$ in degree $p+1$ and zeros everywhere else, corresponds to the Chevalley–Eilenberg algebra

$$\mathrm{CE}(b^{p+1}\mathbb{R}) := \big(\mathbb{R}[g_{p+2}];\ dg_{p+2} = 0\big),$$

where the generator $g_{p+2}$ has degree $p+2$.

Notice that $\mathrm{CE}(b^{p+1}\mathbb{R})$ is the minimal Sullivan model for the rational space $B^{p+2}\mathbb{R}$, reflecting the fact that $b^{p+1}\mathbb{R}$ is the $L_\infty$-algebra corresponding to the $\infty$-group $B^{p+1}\mathbb{R} \simeq \Omega B^{p+2}\mathbb{R}$. In order to amplify this relation between $L_\infty$-algebras and rational homotopy theory, we also write $\mathfrak{l}(X)$, or simply $\mathfrak{l}X$, for the $L_\infty$-algebra whose CE-algebra is a given Sullivan model of finite type for some rational space $X$:

$$\mathfrak{l}(X) = L_\infty\text{-algebra dual to given Sullivan model } (A_X, d_X) \text{ for rationalization of X}$$

i.e.

$$\mathrm{CE}(\mathfrak{l}(X)) := (A_X, d_X).$$

See Appendix A for more details on rational homotopy theory and Sullivan models. For example, with this notation then the rationalized spheres $S^n$ are incarnated as

$$\mathrm{CE}(\mathfrak{l}S^n) = \begin{cases} (\mathbb{R}[g_n], dg_n = 0) & \text{for } n \text{ odd} \\ (\mathbb{R}[g_n, g_{2n-1}], dg_n = 0,\ dg_{2n-1} = g_n \wedge g_n) & \text{for } n > 0 \text{ even.} \end{cases}$$

A convenient feature of the dual picture is the following: if $\mathrm{CE}(\mathfrak{h}) \to \mathrm{CE}(\mathfrak{g})$ is a relative Sullivan algebra, that is, a cofibration in the standard model structure on differential graded commutative algebras (DGCAs), then the corresponding $L_\infty$-morphism $\mathfrak{g} \to \mathfrak{h}$ is a fibration in the model structure whose fibrant objects are $L_\infty$-algebras, due to [10, prop. 4.36, prop. 4.42]. Although relative Sullivan algebras do not exhaust fibrations of $L_\infty$-algebras, they are flexible enough to allow us to realize all the fibrations we will need in the present article as relative Sullivan algebras. See [11] for more on the homotopy theory of $L_\infty$-algebras as a category of fibrant objects.

The model structure whose fibrant objects are $L_\infty$-algebras in [10] is for ordinary $L_\infty$-algebras, not for super $L_\infty$-algebras that we consider here. Nevertheless, the result is readily adapted: A super $L_\infty$-algebra $\mathfrak{g}$ determines a functor $\Lambda \mapsto (\mathfrak{g} \otimes \Lambda)_{\mathrm{even}}$ with values in ordinary $L_\infty$-algebras on the category of finitely generated Grassmann algebras $\Lambda$, and this construction embeds super $L_\infty$-algebras into this functor category. (For super Lie algebras this was observed in [12], see [13] and [14, Cor. 3.3].) Now, by [10, Theorem 4.35], the opposite model structure for ordinary $L_\infty$-algebras is cofibrantly generated, and so a standard argument [15, section 11.6] gives that this functor category inherits the corresponding projective model structure. That is the model structure in which the computations in this paper take place. However, we need to invoke only a bare minimum of model category theory; all we use is the computation of homotopy fibers as ordinary fibers of fibration resolutions. In the following we will find it very useful to succinctly capture results via (commuting) diagrams. We will use the notation $\mathrm{hofib}(\phi)$ to indicate the homotopy fiber of a morphism $\phi$.