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Gaussian and mean curvatures for discrete asymptotic nets

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ABSTRACT

We propose discretisations of Gaussian and mean curvatures of surfaces parametrised in terms of asymptotic coordinates and examine their relevance in the context of integrable discretisations of classical classes of surfaces and their underlying integrable systems. We also record discrete analogues of the classical relation between the Gaussian curvature of hyperbolic surfaces and the torsion of their asymptotic lines.

1. Introduction

It has been established that discrete systems play a fundamental role in the theory of integrable systems and geometric tools to construct and analyse integrable systems have proven to be very useful in this connection. In particular, the intimate relationship between discrete differential geometry and discrete integrable systems has been well documented [1]. Recently, the development of a discrete curvature theory for polyhedral surfaces which respects the integrability properties inherent in its classical continuous counterpart has been a focus of attention. Gaussian and mean curvatures for discrete curvature nets (circular nets) have been proposed in [2] and related to a discrete Steiner formula for parallel nets [3]. A more general curvature theory for discrete surfaces based on mesh parallelity has been put forward in [4]. Moreover, the theory of edge-constraint nets [5] gives rise to the definition of fundamental forms and curvatures for discretisations of surfaces which are parametrised in terms of general coordinates.

Asymptotic coordinates on hyperbolic surfaces are fundamental in classical differential geometry [6] and have been shown to constitute a rich source of integrable connections (see, e.g., [7,8] and references therein). Discrete asymptotic nets, that is, lattices of $\mathbb{Z}^2$ combinatorics with planar stars have proven to be the canonical discrete analogue of surfaces parametrised in terms of asymptotic coordinates [1]. Here, we propose Gaussian and mean curvatures for discrete asymptotic nets and demonstrate their compatibility with a variety of standard discretisations of classical classes of surfaces, namely pseudospherical, minimal and Bianchi surfaces and (generalised) affine spheres, and their underlying integrable systems. Specifically, we show that the Gaussian curvature of discrete pseudospherical surfaces is constant and pairs of conjugate discrete minimal surfaces have vanishing mean curvature in the current sense and coinciding Gaussian curvature. Furthermore, the square root of the negative reciprocal of the vertex Gaussian curvature of discrete Bianchi surfaces is "discrete harmonic" with respect to a weighted discrete d'Alembert operator. In the classical setting, the affine distance of a point on a surface in centro-affine differential geometry may be expressed in terms of the Gaussian curvature of the surface at that point and the Euclidean distance of the corresponding tangent plane to the origin. It is shown that this connection may be transferred to the discrete setting and we demonstrate that the affine distance of discrete affine spheres is constant. We

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also prove that discrete generalised affine spheres are algebraically characterised by the affine distance obeying a "discrete logarithmic" harmonicity condition.

The approach presented here is based on known canonical discrete versions of the classical Lelieuvre formulae and associated Moutard equation [9,10]. These may also be used to determine compact expressions for the discrete torsion of discrete asymptotic lines and its relation to the discrete Gaussian curvature is shown to mirror the well-known connection in the classical setting.

2. Classical Gaussian and mean curvatures

There exist various equivalent definitions of Gaussian and mean curvatures of surfaces Σ in a three-dimensional Euclidean space $\mathbb{R}^3$. For instance, Steiner's formula [11] relates the change of the area of an infinitesimal surface element of a surface Σ to the Gaussian and mean curvatures of Σ as one passes through the associated family of parallel surfaces $\Sigma^{\parallel}$. Thus, if Σ is a surface parametrised by

$$\boldsymbol{r} : \mathbb{R}^2 \to \mathbb{R}^3, \quad (x, y) \mapsto \boldsymbol{r}(x, y)$$

and

$$\boldsymbol{N} = \frac{\boldsymbol{r}_x \times \boldsymbol{r}_y}{|\boldsymbol{r}_x \times \boldsymbol{r}_y|}$$

constitutes an associated unit normal then a parallel surface $\Sigma^{\parallel}$ at (constant) distance μ is represented by

$$\boldsymbol{r}^{\parallel} = \boldsymbol{r} + \mu \boldsymbol{N}$$

so that

$$\boldsymbol{r}_x^{\parallel} \times \boldsymbol{r}_y^{\parallel} = \boldsymbol{r}_x \times \boldsymbol{r}_y + \mu(\boldsymbol{r}_x \times \boldsymbol{N}_y + \boldsymbol{N}_x \times \boldsymbol{r}_y) + \mu^2 \boldsymbol{N}_x \times \boldsymbol{N}_y.$$

Since all terms in the above identity are proportional to $\boldsymbol{N}$, we may, in short-hand notation, formulate Steiner's formula as

$$\frac{\boldsymbol{r}_x^{\parallel} \times \boldsymbol{r}_y^{\parallel}}{\boldsymbol{r}_x \times \boldsymbol{r}_y} = 1 - 2\mu\mathcal{H} + \mu^2\mathcal{K},$$

where the Gaussian and mean curvatures $\mathcal{K}$ and $\mathcal{H}$ of the surface Σ are given by

$$\mathcal{K} = \frac{\boldsymbol{N}_x \times \boldsymbol{N}_y}{\boldsymbol{r}_x \times \boldsymbol{r}_y}, \quad \mathcal{H} = -\frac{\boldsymbol{r}_x \times \boldsymbol{N}_y + \boldsymbol{N}_x \times \boldsymbol{r}_y}{2(\boldsymbol{r}_x \times \boldsymbol{r}_y)}. \tag{1}$$

Here, we are concerned with surfaces which may be parametrised in terms of real asymptotic coordinates, that is, surfaces of negative Gaussian curvature $\mathcal{K}$ (hyperbolic surfaces). A surface Σ is parametrised in terms of asymptotic coordinates if the second fundamental form of Σ is purely off-diagonal, that is,

$$\boldsymbol{r}_x \cdot \boldsymbol{N}_x = \boldsymbol{r}_y \cdot \boldsymbol{N}_y = 0.$$

This is equivalent to stating that the second derivatives $\boldsymbol{r}_{xx}$ and $\boldsymbol{r}_{yy}$ are tangent to Σ. Hence, (x, y) constitute asymptotic coordinates on Σ if and only if the position vector $\boldsymbol{r}$ and the normal $\boldsymbol{N}$ are related by

$$\boldsymbol{r}_x \sim \boldsymbol{N}_x \times \boldsymbol{N}, \quad \boldsymbol{r}_y \sim \boldsymbol{N} \times \boldsymbol{N}_y.$$

It turns out that it is possible to introduce a scaled normal

$$\mathcal{V} = \rho \boldsymbol{N}$$

such that (modulo interchanging x and y)

$$\boldsymbol{r}_x = \mathcal{V}_x \times \mathcal{V}, \quad \boldsymbol{r}_y = \mathcal{V} \times \mathcal{V}_y. \tag{2}$$

The latter are known as Lelieuvre formulae [6]. These encode the fact that surfaces parametrised in terms of asymptotic coordinates are encapsulated in the classical Moutard equation [12]

$$\mathcal{V}_{xy} = h\mathcal{V}. \tag{3}$$

Indeed, the pair (2) is compatible if and only if the compatibility condition $\mathcal{V}_{xy} \times \mathcal{V} = \boldsymbol{0}$ holds. Accordingly, there exists a function h such that the Moutard equation (3) results.

The geometric meaning of the magnitude $|\rho| = |\mathcal{V}|$ of the Lelieuvre normal is revealed by calculating the Gaussian curvature of Σ. Indeed, if we formulate the Lelieuvre formulae as

$$\boldsymbol{r}_x = \rho^2 \boldsymbol{N}_x \times \boldsymbol{N}, \quad \boldsymbol{r}_y = \rho^2 \boldsymbol{N} \times \boldsymbol{N}_y$$