Journal of Geometry and Physics 114 (2017) 153–183

Contents lists available at ScienceDirect

Journal of Geometry and Physics

journal homepage: www.elsevier.com/locate/jgp

Quantum statistical mechanics in arithmetic topology

Matilde Marcolli *, Yujie Xu

Division of Physics, Mathematics, and Astronomy, California Institute of Technology, 1200 E California Blvd, Pasadena, CA 91125, USA

ARTICLE INFO

Article history:
Received 21 August 2016
Received in revised form 27 November 2016
Accepted 28 November 2016
Available online 9 December 2016

Keywords:
Arithmetic topology
Quantum statistical mechanics
Operator algebra
Knots

ABSTRACT

This paper provides a construction of a quantum statistical mechanical system associated to knots in the 3-sphere and cyclic branched coverings of the 3-sphere, which is an analog, in the sense of arithmetic topology, of the Bost–Connes system, with knots replacing primes, and cyclic branched coverings of the 3-sphere replacing abelian extensions of the field of rational numbers. The operator algebraic properties of this system differ significantly from the Bost–Connes case, due to the properties of the action of the semigroup of knots on a direct limit of knot groups. The resulting algebra of observables is a noncommutative Bernoulli product. We describe the main properties of the associated quantum statistical mechanical system and of the relevant partition functions, which are obtained from simple knot invariants like genus and crossing number.

1. Introduction

This paper addresses a question asked to the first author by Masanori Morishita, on the possibility of adapting to 3-manifolds the Bost–Connes construction [1] of a quantum statistical mechanical system associated to the abelian extensions of $\mathbb{Q}$, and its generalizations to number fields [2–5], along the lines of the general "arithmetic topology" program. The latter can be seen as a broad dictionary of analogies between the geometry of knots and 3-manifolds and the arithmetic of number fields, with knots as analogs of primes and 3-manifolds, seen as branched coverings of the 3-sphere, viewed as analogs of number fields. In this paper we answer Morishita's question by providing explicit constructions of quantum statistical mechanical systems associated to (alternating) knots, to knot groups, and to cyclic branched covers of the 3-sphere, with the latter providing our analog of the abelian extensions of $\mathbb{Q}$ in the Bost–Connes construction. The structure of the resulting quantum statistical mechanical systems is different from the Bost–Connes case and it leads to an algebra of observables that can be expressed in the form of a Bernoulli crossed product, of the type studied in noncommutative Bernoulli actions in the theory of factors. We relate the geometry and dynamics of our system to known invariants of knots and 3-manifolds.

1.1. The principle of arithmetic topology

Arithmetic topology originates from insights by John Tate and Michael Artin on topological interpretations of class field theory. The analogy between primes and knots, which is the founding principle of Arithmetic Topology, was first observed by Barry Mazur, David Mumford, and Yuri Manin. The subject developed over the years, with various contributions, such as [6–13], as a powerful guiding principle outlining parallel results and analogies between the arithmetic of number fields and

* Corresponding author.
E-mail addresses: matilde@caltech.edu (M. Marcolli), yujiex@caltech.edu (Y. Xu).

http://dx.doi.org/10.1016/j.geomphys.2016.11.029

the topology of 3-manifolds. The basic analogy sees number fields as analogs of compact oriented 3-manifolds, with $\mathbb{Q}$ playing the role of the 3-sphere S^3. Here the main idea is that, while number fields are finite extensions of $\mathbb{Q}$, ramified at a finite set of primes, all compact oriented 3-manifolds can be described as branched coverings of the 3-sphere, branched along a link. A major point where this analogy does not carry over is the fact that, while the description of a number field as ramified covering of $\mathbb{Q}$ is unique, there are many inequivalent ways of describing 3-manifolds as branched covers of the 3-sphere, branched along knots or links (or more generally embedded graph). While this lack of uniqueness for 3-manifolds can be used to make the construction dynamical, see [14], the same dynamics does not apply to number fields. However, the corresponding analogy between knots and primes, that results from this first analogy between number fields and 3-manifolds, has been very fruitful, leading to many new results, ranging from arithmetic analogs for higher linking numbers [9,10], to arithmetic Chern–Simons theory [8].

Over the past two decades, the connection between number theory and quantum statistical mechanics was also widely explored, starting with early constructions of statistical systems associated to the primes, [15,16], the more refined Bost–Connes system [1] which also involves the Galois theory of abelian extensions of $\mathbb{Q}$, and subsequent generalizations of this construction to arbitrary number fields, obtained in [4] and further studied in [3,17,5,18]. The purpose of the present paper is to recast the Bost–Connes construction in the setting of arithmetic topology, with the semigroup of knots with the connecting sum operation replacing the multiplicative semigroup of positive integers, and the cyclic branched coverings of the 3-spheres replacing the abelian extensions of $\mathbb{Q}$.

1.2. Bost–Connes system

We recall briefly the construction of the Bost–Connes algebra and quantum statistical mechanical system from [1] (see also [19] and §3 of [20]). Consider the group ring $\mathbb{Q}[\mathbb{Q}/\mathbb{Z}]$ with generators $e(r)$ with $r \in \mathbb{Q}/\mathbb{Z}$. The maps $\{\sigma_n\}_{n\in\mathbb{N}}$ given by

$$\sigma_n(e(r)) := e(nr) \tag{1.1}$$

determine an action of the semigroup $\mathbb{N}$ by endomorphisms of the group ring $\mathbb{Q}[\mathbb{Q}/\mathbb{Z}]$. These endomorphisms have partial inverses $\alpha_n : \mathbb{Q}[\mathbb{Q}/\mathbb{Z}] \to \mathbb{Q}[\mathbb{Q}/\mathbb{Z}]$,

$$\alpha_n(e(r)) = \frac{1}{n} \sum_{s:\, ns=r} e(s) \tag{1.2}$$

with $\sigma_n \circ \alpha_n(e(r)) = e(r)$ and $\alpha_n \circ \sigma_n(e(r)) = e_n \cdot e(r)$, with $e_n = n^{-1} \sum_{s:\, ns=0} e(s)$ an idempotent in $\mathbb{Q}[\mathbb{Q}/\mathbb{Z}]$. Thus, one can define the semigroup crossed product. This is the (rational) Bost–Connes algebra $\mathcal{A}_{BC,\mathbb{Q}} = \mathbb{Q}[\mathbb{Q}/\mathbb{Z}] \rtimes \mathbb{N}$ with generators μ_n and $e(r)$ and relations

$$\mu_n^*\mu_n = 1, \quad \mu_n\mu_n^* = e_n, \quad \mu_n\mu_m = \mu_{nm}, \quad \mu_n\mu_m^* = \mu_m^*\mu_n \text{ for } (n,m) = 1, \tag{1.3}$$

$$\mu_n e(r)\mu_n^* = \alpha_n(e(r)), \quad \mu_n^* e(r)\mu_n = \sigma_n(e(r)). \tag{1.4}$$

The complexification $\mathcal{A}_{BC,\mathbb{C}} = \mathcal{A}_{BC,\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{C}$ has a C^*-algebra completion given by the semigroup crossed product $\mathcal{A}_{BC} = C^*(\mathbb{Q}/\mathbb{Z}) \rtimes \mathbb{N}$, with the same generators and relations. The time evolution of the Bost–Connes system is defined by $\sigma_t(\mu_n) = n^{it}\mu_n$ and $\sigma_t(e(r)) = e(r)$. The algebra $\mathcal{A}_{BC}$ has representations on the Hilbert space $\ell^2(\mathbb{N})$, parameterized by the choice of an element $u \in \hat{\mathbb{Z}}^*$, of the form

$$\pi_u(e(r))\epsilon_m = u(r)^m\,\epsilon_m, \quad \pi_u(\mu_n)\epsilon_m = \epsilon_{nm}, \tag{1.5}$$

where $u(r)$ is a root of unity in $\mathbb{C}$ determined by the embedding of $\mathbb{Q}/\mathbb{Z} \hookrightarrow \mathbb{C}$ specified by the choice of $u \in \hat{\mathbb{Z}}^*$, where we identify $\hat{\mathbb{Z}} = \mathrm{Hom}(\mathbb{Q}/\mathbb{Z}, \mathbb{Q}/\mathbb{Z})$.

Given a pair $(\mathcal{A}, \sigma)$ of a C^*-algebra and a time evolution $\sigma : \mathbb{R} \to \mathrm{Aut}(\mathcal{A})$, a KMS$_\beta$ state for $(\mathcal{A}, \sigma)$ is a continuous linear functional $\varphi_\beta : \mathcal{A} \to \mathbb{C}$ satisfying normalization $\varphi_\beta(1) = 1$ and positivity $\varphi_\beta(a^*a) \geq 0$ (that is, a *state* on $\mathcal{A}$) such that, for all $a, b \in \mathcal{A}$ there is a function $F_{a,b}(z)$ that is holomorphic on the strip $\mathcal{I}_\beta = \{z \in \mathbb{C} \,:\, 0 < \Im(z) < \beta\}$ and continuous on the boundary $\partial\mathcal{I}_\beta$ of the strip, such that

$$F_{a,b}(t) = \varphi_\beta(a\sigma_t(b)), \quad F_{a,b}(t + i\beta) = \varphi(\sigma_t(b)a). \tag{1.6}$$

In other words, the failure of a KMS$_\beta$ to be a trace is measured by interpolation by a holomorphic function.

The KMS states of the Bost–Connes system $(\mathcal{A}_{BC}, \sigma)$ are completely classified and given by the following list of cases (see [1]):

- for every $0 < \beta \leq 1$ there is a unique KMS$_\beta$ state φ_β determined by

  $$\varphi_\beta\left(e\left(\frac{a}{b}\right)\right) = \frac{f_{-\beta+1}(b)}{f_1(b)}$$

  where $f_k(b) = \sum_{d|b} \mu(d)(b/d)^k$, with μ the Möbius function;