Journal of Geometry and Physics 114 (2017) 260–275

Contents lists available at ScienceDirect

Journal of Geometry and Physics

journal homepage: www.elsevier.com/locate/jgp

Second Yamabe constant on Riemannian products

Guillermo Henry

Departamento de Matemática, FCEyN, Universidad de Buenos Aires, Ciudad Universitaria, Pab. I., C1428EHA, Buenos Aires, Argentina

ARTICLE INFO

Article history:
Received 29 February 2016
Received in revised form 22 November 2016
Accepted 28 November 2016
Available online 14 December 2016

MSC:
53C21

Keywords:
Second Yamabe constant
Yamabe equation
Nodal solutions

ABSTRACT

Let (M^m, g) be a closed Riemannian manifold ($m \geq 2$) of positive scalar curvature and (N^n, h) any closed manifold. We study the asymptotic behaviour of the second Yamabe constant and the second N-Yamabe constant of $(M \times N, g + th)$ as t goes to $+\infty$. We obtain that $\lim_{t\to+\infty} Y^2(M \times N, [g + th]) = 2^{\frac{2}{m+n}} Y(M \times \mathbb{R}^n, [g + g_e])$. If $n \geq 2$, we show the existence of nodal solutions of the Yamabe equation on $(M \times N, g + th)$ (provided t large enough). When s_g is constant, we prove that $\lim_{t\to+\infty} Y^2_N(M \times N, g + th) = 2^{\frac{2}{m+n}} Y_{\mathbb{R}^n}(M \times \mathbb{R}^n, g + g_e)$. Also we study the second Yamabe invariant and the second N-Yamabe invariant.

1. Introduction

Let (W^k, G) be a closed Riemannian manifold of dimension $k \geq 3$ with scalar curvature s_G. The Yamabe functional $J : C^\infty(W) - \{0\} \longrightarrow \mathbb{R}$ is defined by

$$J(u) := \frac{\int_W a_k |\nabla u|_G^2 + s_G u^2 dv_G}{\|u\|_{p_k}^2}$$

where $a_k := 4(k-1)/(k-2)$ and $p_k := 2k/(k-2)$.

The infimum of the Yamabe functional over the set of smooth functions of W, excluding the zero function, is a conformal invariant and it is called the Yamabe constant of W in the conformal class of G (which we are going to denote by $[G]$):

$$Y(W, [G]) = \inf_{u \in C^\infty(W) - \{0\}} J(u).$$

Recall that the conformal Laplacian operator of (W, G) is

$$L_G := a_k \Delta_G + s_G,$$

where Δ_G is the negative Laplacian, i.e., $\Delta_{g_e} u = -\sum_{i=1}^n \frac{\partial^2 u}{\partial x_i^2}$ in the Euclidean space $(\mathbb{R}^n, g_e)$.

The celebrated Yamabe problem states that in any conformal class of a closed Riemannian manifold (of dimension at least 3) there exists a Riemannian metric with constant scalar curvature. This was proved in a series of articles by Yamabe [1], Trudinger [2], Aubin [3], and Schoen [4]. Actually, they proved that the Yamabe constant is attained by a smooth positive

E-mail address: ghenry@dm.uba.ar.

http://dx.doi.org/10.1016/j.geomphys.2016.11.025
0393-0440/

function u_{min}. It can be seen that a function u_{cp} is a critical point of the Yamabe functional if and only if it solves the so called Yamabe equation

$$L_G(u_{cp}) = \lambda |u_{cp}|^{p_k-2} u_{cp} \tag{1}$$

for $\lambda = J(u_{cp})/\|u_{cp}\|_{p_k}^{p_k-2}$. Recall that if $\tilde{G}$ belongs to $[G]$, then

$$L_G(u) = s_{\tilde{G}} u^{p_k-1}$$

where u is the positive smooth function that satisfies $\tilde{G} = u^{p_k-2}G$. Therefore, $G_{u_{min}} := u_{min}^{p_k-2}G$ must be a metric of constant scalar curvature.

The solution of the Yamabe problem provides a positive smooth solution of the Yamabe equation. Actually, as we pointed out, there is a one to one relationship between the Riemannian metrics with constant scalar curvature in $[G]$ and positive solutions of the Yamabe equation.

Nevertheless, in order to understand the set of solutions of the Yamabe equation, it seems important to study the nodal solutions, i.e., a sign changing solution of (1). In the last years several authors have addressed the question about the existence and multiplicity of nodal solutions of the Yamabe equation: Hebey and Vaugon [5], Holcman [6], Jourdain [7], Djadli and Jourdain [8], Ammann and Humbert [9], Petean [10], El Sayed [11] among others.

Let

$$\lambda_1(L_G) < \lambda_2(L_G) \leq \lambda_3(L_G) \leq \cdots \nearrow +\infty$$

be the sequence of eigenvalues of L_G, where each eigenvalue appears repeated according to its multiplicity. It is well known that it is an increasing sequence that tends to infinity.

When $Y(W, [G]) \geq 0$, it is not difficult to see that

$$Y(W, [G]) = \inf_{\tilde{G}\in[G]} \lambda_1(L_{\tilde{G}}) vol(W, \tilde{G})^{\frac{2}{k}},$$

where $vol(W, \tilde{G})$ is the volume of $(W, \tilde{G})$.

In [9], Ammann and Humbert introduced the lth Yamabe constant. This constant is defined by

$$Y^l(W, [G]) := \inf_{\tilde{G}\in[G]} \lambda_l(L_{\tilde{G}}) vol(W, \tilde{G})^{\frac{2}{k}}.$$

Like the Yamabe constant, the lth Yamabe constant is a conformal invariant.

They showed that the second Yamabe constant of a connected Riemannian manifold with non-negative Yamabe constant is never achieved by a Riemannian metric. Nevertheless, if we enlarge the conformal class, allowing generalized metrics (i.e., metrics of the form $u^{p_k-2}G$ with $u \in L^{p_k}(W)$, $u \geq 0$, and u does not vanish identically), under some assumptions on (W, G), the second Yamabe constant is achieved ([9], Corollary 1.7). Moreover, if $Y^2(W, G) > 0$, they proved that if a generalized metric $\tilde{G}$ realizes the second Yamabe constant, then it is of the form $|w|^{p_k-2}G$ with $w \in C^{3,\alpha}(W)$ a nodal solution of the Yamabe equation. If $Y^2(W, G) = 0$ and is attained, then any eigenfunction corresponding to the second eigenvalue of L_G is a nodal solution.

Therefore, if we know that the second Yamabe constant is achieved, we have a nodal solution of the Yamabe equation. However, this is not the general situation. There exist some Riemannian manifolds for which the second Yamabe constant is not achieved, even by a generalized metric. For instance, (S^k, g_0^k) where g_0^k is the round metric of curvature 1 (cf. [9], Proposition 5.3).

Let (M, g) and (N, h) be closed Riemannian manifolds and consider the Riemannian product $(M \times N, g+h)$. We define the N-Yamabe constant as the infimum of the Yamabe functional over the set of smooth functions, excluding the zero function, that depend only on N:

$$Y_N(M \times N, g+h) := \inf_{u\in C^\infty(N)-\{0\}} J(u).$$

Clearly, $Y(M\times N, g+h) \leq Y_N(M\times N, g+h)$. The N-Yamabe constant is not a conformal invariant, but it is scale invariant. It was first introduced by Akutagawa, Florit, and Petean in [12], where they studied, among other things, its behaviour on Riemannian products of the form $(M \times N, g + th)$ with $t > 0$.

Actually, the infimum of J over $C^\infty(N) - \{0\}$ is a minimum, and it is achieved by a positive smooth function.

When the scalar curvature of the product is constant, the critical points of the Yamabe functional restricted to $C^\infty(N) - \{0\}$, satisfy the Yamabe equation, and thereby, also satisfy the subcritical Yamabe equation (recall that $p_{m+n} < p_n$). Hence, if $Y_N(M \times N, g + h) = J(u)$, then the metric $G = u^{p_{m+n}-2}(g + h) \in [g + h]$ has constant scalar curvature. When $s_{g+h} \leq 0$, the Yamabe constant of $(M \times N, g + h)$ is nonpositive, and in this situation, there is essentially only one metric of constant scalar curvature, the metric $g + h$. Therefore, this case is not interesting.

It seems important to consider the N-Yamabe constant because in some cases the minimizer (or some minimizers) of the Yamabe functional depends only on one of the variables of the product. For instance, it was proved by Kobayashi in [13] and Schoen in [14] that the minimizer of the Yamabe functional on $(S^n \times S^1, g_0^n + tg_0^1)$ depends only on S^1. Also, this might be the