Journal of Geometry and Physics 114 (2017) 276–290

Contents lists available at ScienceDirect

Journal of Geometry and Physics

journal homepage: www.elsevier.com/locate/jgp

# Geometry of warped product immersions of Kenmotsu space forms and its applications to slant immersions

Akram Ali [a], Pişcoran Laurian-Ioan [b,*]

[a] Institute of Mathematical Sciences, Faculty of Science, University of Malaya, 50603, Kuala Lumpur, Malaysia

[b] Department of Mathematics and Computer Science, Victoriei 76, North University Center of Baia Mare Technical University of Cluj Napoca, 430122, Baia Mare, Romania

ARTICLE INFO

*Article history:*
Received 1 October 2016
Received in revised form 29 November 2016
Accepted 6 December 2016
Available online 15 December 2016

*MSC:*
primary 53C40
secondary 53C20 53C42
53B25
53Z05 53C40
58J60

*Keywords:*
Warped products
Semi-slant immersions
Connected
Compact Riemannian submanifolds
Kenmotsu space forms
Hamiltonian
Kinetic energy

ABSTRACT

In this paper, some relations among the second fundamental form which is an extrinsic invariant, Laplacian of the warping function and constant sectional curvature of a warped product semi-slant submanifold of a Kenmotsu space form and its totally geodesic and totally umbilical submanifolds are described from the exploitation of the Gauss equation instead of the Codazzi equation in the sense of Chen's studies in (2003). These relations provide us an approach to the classifications of equalities by the following case studied of Hasegawa and Mihai (2003). These are exemplified by the classifications of the totally geodesic and totally umbilical submanifolds. Moreover, we provide some applications of the inequality case by using the harmonicity of the smooth warping functions. In particular, we prove the triviality of connected, compact warped product semi-slant manifolds isometrically immersed into a Kenmotsu space form using Hamiltonian, Hessian, and the Kinetic energy of the warped function. Further, we generalize some results for contact CR-warped products in a Kenmotsu space form.

© 2016 Elsevier B.V. All rights reserved.

## 1. Introduction and motivations

The study of warped product submanifolds of different ambient manifold with constant sectional curvature has been an active field of research in Riemannian submanifolds theory for years. Especially, Chen in [1,2] inaugurated the idea of CR-warped product submanifolds in Kaehler manifolds and he constructed the first inequality for the second fundamental form which is related to the squared norm of warping function. So far, these types of inequalities represent one of the most fundamental problems regarding the warped product submanifolds. Therefore, many authors have discussed a lot of interesting geometric properties of these warped product immersions types (for examples see [3–14] and [15–20] for more details). The beautiful theory of the warped product submanifolds has rich applications in both Riemannian geometry and semi-Riemannian geometry. Until now, the concept of warped product has been playing a crucial role in the theory of general relativity which provides the best mathematical model of the universe. The warped product model was successfully applied in general relativity and semi-Riemannian geometry in the direction to build basic cosmological models such as the Robertson-Walker spacetime, the Friedmann cosmological model and the standard static spacetime [21].

* Corresponding author.
*E-mail addresses:* akramali133@gmail.com (A. Ali), plaurian@yahoo.com (P. Laurian-Ioan).

http://dx.doi.org/10.1016/j.geomphys.2016.12.001
0393-0440/© 2016 Elsevier B.V. All rights reserved.

It is well known that the branch of warped product manifolds comes into light after initiated by Bishop and O'Neill [22] as a simple generalization of Riemannian product manifolds. According to them, these manifolds are defined as follows:

**Definition 1.1.** Let $(N_1, g_1)$ and $(N_2, g_2)$ be two Riemannian manifolds and $f : N_1 \to (0, \infty)$, a positive differentiable function on $N_1$. Consider the product manifold $N_1 \times N_2$ with its canonical projections $\gamma_1 : N_1 \times N_2 \to N_1$, $\gamma_2 : N_1 \times N_2 \to N_2$ and the projection maps given by $\gamma_1(t, s) = t$ and $\gamma_2(t, s) = s$ for every $l = (t, s) \in N_1 \times N_2$. The warped product $M = N_1 \times_f N_2$ is the product manifold $N_1 \times N_2$ equipped with the Riemannian structure such that

$$\|X\|^2 = \|\gamma_1 * (U)\|^2 + f^2(\gamma_1(t))\|\pi_2 * (X)\|^2, \tag{1.1}$$

for any tangent vector $X \in \mathscr{X}(T_tM)$, where $*$ is the symbol of differential maps and $g = g_1 + f^2 g_2$ is the Riemannian metric on $M$. Thus the function $f$ is called a warping function of $M$.

The following lemma can be seen as a direct consequence of the warped product manifolds:

**Lemma 1.1** *([22]). Let $M = N_1 \times_f N_2$ be a warped product manifold. For any $X, Y \in \mathscr{X}(TN_1)$ and $Z, W \in \mathscr{X}(TN_2)$, we have*

(i) $\nabla_X Y \in \mathscr{X}(TN_1)$,
(ii) $\nabla_Z X = \nabla_X Z = (X \ln f)Z$,
(iii) $\nabla_Z W = \nabla'_Z W - g(Z, W)\nabla \ln f$,

*where $\nabla$ is the Levi-Civita connection on $M$ and $\nabla \ln f$ is the gradient of $\ln f$ which is defined as $g(\nabla \ln f, U) = U \ln f$.*

Now we will give the following definition based on the above lemma,

**Definition 1.2.** A warped product manifold $M = N_1 \times_f N_2$ is said to be *trivial* if the warping function $f$ is constant or in other words, a warped product manifold $M = N_1 \times_f N_2$ is called simply a Riemannian product if $f$ is constant function on $M$.

**Definition 1.3.** If $M = N_1 \times_f N_2$ is a warped product manifold, then $N_1(N_2)$ are called totally geodesics (or totally umbilical submanifold) of $M$, respectively.

By the isometrically embedding theorem of J. F. Nash [23] we know that every Riemannian manifold can be isometrically immersed into an Euclidean space with sufficiently high dimension. Afterward, followed the concept of Nolker [24], Chen in [7] developed a sharp inequality under the name of another general inequality in a CR-warped product $N_T^h \times_f N_\perp^p$ in a complex space form $M^m(4c)$ with a holomorphic constant sectional curvature $4c$ by means of Codazzi equation satisfying the relation

$$\|\sigma\|^2 \leq 2p\bigg(\|\nabla \ln f\|^2 + \Delta(\ln f) + 4hc\bigg),$$

where $h = \dim_C N_T$, $p = \dim N_\perp$ and $\sigma$ is the second fundamental form. Therefore, it is called Chen's second inequality of the second fundamental form. Inspired by these studies, other geometers in [8,10,11,25] obtained some sharp inequalities for the squared norm of the second fundamental form, which is an extrinsic invariant, in terms of the warping function for the contact CR-warped products isometrically immersed in both a Sasakian space form and a Kenmotsu space form using the same techniques by taking equation of Codazzi. Some classifications of contact CR-warped products in spheres which satisfy the equality cases identically are given.

After the slant immersion concept was introduced in [26,27] and corresponding slant curve studied in [28,29], this impose significant restrictions on the geometry of its generalization. Therefore, the warped product semi-slant submanifold might be regarded as the simplest generalization of CR-warped product submanifolds. For instance, the warped product semi-slant submanifold does not admits non-trivial warped product in some ambient manifolds (see [30,31]). Moreover, in [6], M. Atceken studied the non existence of warped product semi-slant submanifolds of a Kenmotsu manifold for which the structure vector $\xi$ is tangent to fiber. Meanwhile, the existence of warped product semi-slant submanifold of a Kenmotsu manifold of the form $M = N_T \times_f N_\theta$ or $M = N_\theta \times_f N_T$, in the case that the structure vector field $\xi$ is tangent to $N_T$ or $N_\theta$ respectively, has been proved in [32,33]. They have obtained lots of examples on the existence of the warped product semi-slant products in a Kenmotsu manifold and derived many general inequalities for the second fundamental form in terms of warping functions.

Consequently, there was a difficulty to describe the differential geometric properties and to establish a relation between the squared norm of the second fundamental form and warping functions in terms of slant immersions by using Codazzi. Applying a new method under assumption of the Gauss equation instead of the Codazzi equation, we establish a sharp general inequality for the warped product semi-slant submanifolds which are isometrically immersed into a Kenmotsu space form as a generalization of the contact CR-warped products. To do these, we consider a non-trivial warped product semi-slant submanifold of the type $M^n = N_T^{n_1} \times_f N_\theta^{n_2}$ in a Kenmotsu manifold and obtain the following result.

**Theorem 1.1.** *Assume that $\chi : M^n = N_T^{n_1} \times_f N_\theta^{n_2} \to \widetilde{M}^{2m+1}(c)$ is an isometric immersion of a warped product semi-slant $N_T^{n_1} \times_f N_\theta^{n_2}$ into a Kenmotsu space form $\widetilde{M}^{2m+1}(c)$. Then*

Download English Version:

https://daneshyari.com/en/article/5500089

Download Persian Version:

https://daneshyari.com/article/5500089

Daneshyari.com