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Formal Killing fields for minimal Lagrangian surfaces in complex space forms

Joe S. Wang

Seoul, South Korea

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ABSTRACT

The differential system for minimal Lagrangian surfaces in a $2_{\mathbb{C}}$-dimensional, non-flat, complex space form is an elliptic integrable system defined on the Grassmann bundle of oriented Lagrangian 2-planes. This is a 6-symmetric space associated with the Lie group $SL(3,\mathbb{C})$, and the minimal Lagrangian surfaces arise as the primitive maps. Utilizing this property, we derive the inductive differential algebraic formulas for a pair of the formal loop algebra $\mathfrak{sl}(3,\mathbb{C})[[\lambda]]$-valued canonical formal Killing fields. For applications, (a) we give a complete classification of the (pseudo) Jacobi fields for the minimal Lagrangian system, (b) we obtain an infinite sequence of conservation laws from the components of the canonical formal Killing fields.

1. Introduction

1.1. Minimal Lagrangian surfaces

In relation to the developments in string theory, special Lagrangian submanifolds in Calabi–Yau manifolds have received much attention recently, [1–11]. When the ambient Calabi–Yau manifold is $\mathbb{C}^{m+1}$ (flat case), the link of a special Lagrangian cone in the unit sphere $S^{2m+1} \subset \mathbb{C}^{m+1}$ is called a special Legendrian submanifold. Under the Hopf map $S^{2m+1} \to \mathbb{CP}^m$, a special Legendrian submanifold corresponds to a minimal Lagrangian submanifold in $\mathbb{CP}^m$.

In the 2-dimensional case, Schoen and Wolfson gave a variational analysis of the area minimizing (Hamiltonian stationary) Lagrangian surfaces in a Kähler surface, [12]. They proved the existence of an area minimizer in a given Lagrangian homology class, possibly with the certain admissible conical singularities. In the absence of the singularities, the area minimizer is minimal Lagrangian. Haskins and Kapouleas gave a gluing construction of the compact high-genus special Legendrian surfaces in the 5-sphere, [8]. For the integrable system aspects of the theory on the minimal Lagrangian tori in $\mathbb{CP}^2$, we refer to [13] and the references therein.

In case of the hyperbolic complex space form $\mathbb{CH}^2$, Loftin and McIntosh gave an analysis of the minimal Lagrangian surfaces in relation to the surface group representations in the Lorentzian special unitary group $SU(1,2)$, [14].

E-mail address: jswang12@gmail.com.

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1.2. Elliptic Tzitzeica equation

The partial differential equation which locally describes a minimal Lagrangian surface (away from the zero divisor of Hopf differential, Section 2.3) in a $2_{\mathbb{C}}$-dimensional[1] complex space form is the elliptic Tzitzeica equation (4.38). It is a rank-1 Toda field equation, which is a well known example of elliptic integrable equation, and the infinite sequence of Jacobi fields and conservation laws were completely determined in [15,16].

The elliptic Tzitzeica equation admits a $\mathfrak{sl}(3,\mathbb{C})$-valued Lax representation, i.e., a loop algebra $\mathfrak{sl}(3,\mathbb{C})[\lambda^{-1},\lambda]$-valued[2] 1-form ψ_λ which satisfies the Maurer–Cartan equation

$$\mathrm{d}\psi_\lambda + \psi_\lambda \wedge \psi_\lambda = 0.$$

The main idea of construction in [16] is to consider the associated Killing field equation with respect to ψ_λ,

$$\mathrm{d}\mathbf{X}_\lambda + [\psi_\lambda, \mathbf{X}_\lambda] = 0, \tag{1.1}$$

where the Killing field $\mathbf{X}_\lambda$ takes values in $\mathfrak{sl}(3,\mathbb{C})[[\lambda^{-1},\lambda]]$. When Eq. (1.1) is expanded as a formal series in the spectral parameter λ, it exhibits a pair of 6-step recursion relations embedded in the infinite jet space of the elliptic Tzitzeica equation. Via an analysis of the conservation laws as characteristic cohomology, a repeated application of the recursion generates the infinite sequence of Jacobi fields and conservation laws.

1.3. Purpose

In the previous work [17] on constant mean curvature (CMC) surfaces, we gave an interpretation of the classical work [18] by Pinkall and Sterling via the associated loop algebras and showed that a CMC surface in a 3-dimensional Riemannian space form admits a loop algebra $\mathfrak{sl}(2,\mathbb{C})[[\lambda]]$-valued canonical formal Killing field. As a consequence, the infinite sequence of higher-order Jacobi fields and conservation laws were determined from the components of the formal Killing field.

We claim that the results of [17] are true for the integrable matrix Lax equations in general; the existence of canonical formal Killing fields is likely a universal property.

In the present paper, we verify this claim for the differential system for minimal Lagrangian surfaces in a $2_{\mathbb{C}}$-dimensional, non-flat, complex space form. We show that such a minimal Lagrangian surface admits a pair of the loop algebra $\mathfrak{sl}(3,\mathbb{C})[[\lambda]]$-valued canonical formal Killing fields, Theorems 5.19 and 5.21. This can be considered as an extension and refinement of the results for the elliptic Tzitzeica equation in [15,16].

The main idea of construction is similar to [16,17]. We apply the recursion relations from the formal Killing field equation to obtain the inductive differential algebraic formulas for the canonical formal Killing fields.

1.3.1. $\mathfrak{sl}(2,\mathbb{C})$ vs. $\mathfrak{sl}(3,\mathbb{C})$

The underlying Lie algebra for the analysis of the CMC surfaces in [17] is $\mathfrak{sl}(2,\mathbb{C})$, which has rank 1. For the minimal Lagrangian surfaces, it is $\mathfrak{sl}(3,\mathbb{C})$, which has rank 2. In hindsight, this accounts for the appearance of two canonical Killing fields.

More generally, consider for example the two dimensional harmonic map equation into the compact special unitary group SU_{N+1}, which is also a well known integrable matrix Lax equation. In this case, it is known that there exists an $N = \mathrm{rank}(\mathrm{SU}_{N+1})$-dimensional space of canonical formal Killing fields.

1.4. Main results

See Notation 5.1 for the relevant notations for loop algebras.

1.4.1. A pair of canonical formal Killing fields

We give a construction of a pair of $\mathfrak{sl}(3,\mathbb{C})[[\lambda]]$-valued canonical formal Killing fields by explicit inductive differential algebraic formulas, Theorems 5.19 and 5.21.

The construction relies on a 6-step recursion process, which involves two steps at which one needs to solve certain $\partial_{\overline{\xi}}$-equations, see Section 5.1.4. The idea is to bypass this problem by imposing the compatible algebraic constraints on the characteristic polynomials of the formal Killing fields.

As a corollary, for example, it implies that a minimal Lagrangian torus in $\mathbb{CP}^2$ admits a pair of canonical polynomial Killing fields, and hence a pair of spectral curves.

[1] Here "$2_{\mathbb{C}}$-dimensional" means "2 complex dimensional".
[2] See Notation 5.1.