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# On local invariants of singular symplectic forms[☆]

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ABSTRACT

We find a complete set of local invariants of singular symplectic forms with the structurally stable Martinet hypersurface on a $2n$-dimensional manifold. In the $\mathbb{C}$-analytic category this set consists of the Martinet hypersurface $\Sigma_2$, the restriction of the singular symplectic form $\omega$ to $T\Sigma_2$ and the kernel of $\omega^{n-1}$ at the point $p \in \Sigma_2$. In the $\mathbb{R}$-analytic and smooth categories this set contains one more invariant: the canonical orientation of $\Sigma_2$. We find the conditions to determine the kernel of $\omega^{n-1}$ at $p$ by the other invariants. In dimension 4 we find sufficient conditions to determine the equivalence class of a singular symplectic form-germ with the structurally smooth Martinet hypersurface by the Martinet hypersurface and the restriction of the singular symplectic form to it. We also study the singular symplectic forms with singular Martinet hypersurfaces. We prove that the equivalence class of such singular symplectic form-germ is determined by the Martinet hypersurface, the canonical orientation of its regular part and the restriction of the singular symplectic form to its regular part if the Martinet hypersurface is a quasi-homogeneous hypersurface with an isolated singularity.



## 1. Introduction

A closed differential 2-form $\omega$ on a $2n$-dimensional smooth manifold $M$ is **symplectic** if $\omega$ is nondegenerate. This means that $\omega$ satisfies the following condition

$$\omega^n|_p = \omega \wedge \cdots \wedge \omega|_p \neq 0, \ \text{for } p \in M. \tag{1.1}$$

A closed differential 2-form $\omega$ on a $2n$-dimensional smooth manifold $M$ is called a **singular symplectic** form if the set of points where $\omega$ does not satisfy (1.1):

$$\{p \in M : \omega^n|_p = 0\} \tag{1.2}$$

is nowhere dense. We denote the set (1.2) by $\Sigma_2(\omega)$ or $\Sigma_2$. It is called the **Martinet hypersurface**.

Singular symplectic forms appear naturally if one studies classification of germs of submanifolds of a symplectic manifold. By Darboux–Givental theorem ([1], see also [2]) germs of submanifolds of the symplectic manifold are symplectomorphic if and only if the restrictions of the symplectic form to them are diffeomorphic. This theorem reduces the problem of local classification of generic submanifolds of the symplectic manifold to the problem of local classification of singular symplectic forms.

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Singular symplectic forms can be applied in thermodynamics: in the modeling the absolute zero temperature region (see [3]). The first occurring singularity of singular symplectic forms is the Martinet singularity of type $\Sigma_{20}$. It has the following local normal form in coordinates $(x_1, y_1, \ldots, x_n, y_n)$ on $\mathbb{R}^{2n}$ [4]

$$\omega = x_1 dx_1 \wedge dy_1 + \sum_{i=2}^{n} dx_i \wedge dy_i. \tag{1.3}$$

The Martinet hypersurface of the Martinet singular symplectic form of type $\Sigma_{20}$ is smooth and the restriction of this singular form to the Martinet hypersurface has the maximal rank. This singular symplectic form gives a fine link between the thermodynamical postulate of positivity of absolute temperature and the stability of an applicable structure of thermodynamics [5].

By the classical Darboux theorem all symplectic forms on $M$ are locally diffeomorphic i.e. there exists a diffeomorphism-germ of $M$ mapping the germ of one symplectic form to the germ of the other.

This is no longer true if we consider singular symplectic forms. It is obvious that if germs of singular symplectic forms $\omega_1$ and $\omega_2$ are diffeomorphic then the germs of corresponding Martinet hypersurfaces $\Sigma_2(\omega_1)$ and $\Sigma_2(\omega_2)$ must be diffeomorphic and the restrictions of germs of singular symplectic forms $\omega_1$ and $\omega_2$ to the regular parts of $\Sigma_2(\omega_1)$ and $\Sigma_2(\omega_2)$ respectively must be diffeomorphic too.

In this paper we study if the inverse theorem is valid:

*Do the Martinet hypersurface $\Sigma_2$ and the restriction of $\omega$ to the regular part of $\Sigma_2$ form a complete set of invariants of $\omega$?*

Because our consideration is local, we may assume that $\omega$ is a $\mathbb{K}$-analytic or smooth closed 2-form-germ at 0 on $\mathbb{K}^{2n}$ for $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$.

Then $\omega^n = f\Omega$, where $f$ is a function-germ at 0 and $\Omega$ is the germ at 0 of a volume form on $\mathbb{K}^{2n}$. The Martinet hypersurface has the form $\Sigma_2 = \{f = 0\}$ and it is a called **structurally smooth at** 0 if $f(0) = 0$ and $df_0 \neq 0$. Then $\Sigma_2$ is a smooth hypersurface-germ. In dimension 4 such situation is generic.

Local invariants of singular contact structures were studied in [6] and [7]. B. Jakubczyk and M. Zhitomirskii show that local $\mathbb{C}$-analytic singular contact structures on $\mathbb{C}^3$ with structurally smooth Martinet hypersurfaces $S$ are diffeomorphic if their Martinet hypersurfaces and restrictions of singular structures to them are diffeomorphic. In the $\mathbb{R}$-analytic category a complete set of invariants contains, in general, one more independent invariant. It is a canonical orientation on the Martinet hypersurface. The same is true for smooth local singular contact structures $P = (\alpha)$ on $\mathbb{R}^3$ provided $\alpha|_S$ is either not flat at 0 or $\alpha|_S = 0$. The authors also study local singular contact structures in higher dimensions. They find more subtle invariants of a singular contact structure $P = (\alpha)$ on $\mathbb{K}^{2n+1}$: a line bundle $L$ over the Martinet hypersurface $S$, a canonical partial connection $\Delta_0$ on the line bundle $L$ at $0 \in \mathbb{K}^{2n+1}$ and a 2-dimensional kernel $ker(\alpha \wedge (d\alpha)^n)|_0$. They also consider the more general case when $S$ has singularities.

For the first occurring singularities of singular symplectic forms on a 4-dimensional manifold the answer for the above question follows from Martinet's normal forms (see [4], [8,9]). In fact it is proved that the Martinet hypersurface $\Sigma_2$ and a characteristic line field on $\Sigma_2$ (i.e. $\{X$ is a smooth vector field : $X \rfloor (\omega|_{T\Sigma_2}) = 0\}$) form a complete set of invariants of generic singularities of singular symplectic forms on a 4-dimensional manifold.

In this paper we show that a complete set of invariants for $\mathbb{C}$-analytic singular symplectic form-germs on $\mathbb{C}^{2n}$ with structurally smooth Martinet hypersurfaces consists of the Martinet hypersurface, the pullback of the singular form-germ $\omega$ to it and the 2-dimensional kernel of $\omega^{n-1}|_0$ (Theorem 2.2). The same is true for local $\mathbb{R}$-analytic and smooth singular symplectic forms on $\mathbb{R}^{2n}$ with structurally smooth Martinet hypersurfaces if we include in the set of invariants the canonical orientation of the Martinet hypersurface (Theorem 2.3).

In Section 4 we also prove that an equivalence class of a smooth or $\mathbb{K}$-analytic singular symplectic form-germ $\omega$ on $\mathbb{K}^{2n}$ with the structurally smooth Martinet hypersurface is determined only by the Martinet hypersurface, its canonical orientation (only if $\mathbb{K} = \mathbb{R}$) and the pullback of the singular form-germ to it if the dimension of a vector space generated by the coefficients of the 1-jet at 0 of $(\omega|_{T\Sigma_2})^{n-1}$ is equal to 2.

In Section 5 we consider singular symplectic forms on $\mathbb{K}^4$ with structurally smooth Martinet hypersurfaces. We show that an equivalence class of a smooth or $\mathbb{K}$-analytic singular symplectic form $\omega$ on $\mathbb{K}^4$ with a structurally smooth Martinet hypersurface is determined only by the Martinet hypersurface and the pullback of the singular form to it if the two generators of the ideal generated by coefficients of $\omega|_{T\Sigma_2}$ form a regular sequence.

In $\mathbb{C}$-analytic category we prove the same result for a wider class of singular symplectic forms. The analogous result in $\mathbb{R}$-analytic category requires the assumption on the canonical orientation. The preliminary versions of results of Section 5 were presented in [10] (Theorems 5.1, 5.2, Proposition 5.3).

We also consider singular symplectic forms with singular Martinet hypersurfaces. We prove that if the Martinet hypersurface of a singular symplectic form-germ is a quasi-homogeneous hypersurface-germ with an isolated singularity then the complete set of local invariants of this singular form consists of the canonical orientation of the regular part of the Martinet hypersurface (for $\mathbb{K} = \mathbb{R}$ only) and the restriction of the singular form to the regular part of the Martinet hypersurface.