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Energy identity for harmonic maps into standard stationary Lorentzian manifolds

Xiaoli Han [a,*], Liang Zhao [b], Miaomiao Zhu [c]

[a] Department of Mathematical Sciences, Tsinghua University, Beijing 100084, China

[b] School of Mathematical Sciences, Beijing Normal University, Laboratory of Mathematics and Complex Systems, Ministry of Education, Beijing 100875, China

[c] School of Mathematical Sciences, Shanghai Jiao Tong University, Dongchuan Road 800, Shanghai 200240, China

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ABSTRACT

For a harmonic map from a closed Riemann surface into a standard stationary Lorentzian manifold, we prove that its Hopf differential is holomorphic. Moreover, we prove that for a sequence of such maps with their energy uniformly bounded, the Lorentzian energy identity holds during the blow-up process.

1. Introduction

The theory of harmonic maps from Riemann surfaces into compact Riemannian manifolds plays an important role in geometric analysis and it has been extensively studied in the literature. Due to their conformal invariance, a lot of powerful analytical tools (for instance, integrability by compensation and blow-up analysis) have been developed to study the regularity problem (see e.g. [1–4]) and the compactness problem (see e.g. [5–8]). However, these classical methods rely on the fact that the target is compact and Riemannian. Motivated by the correspondence between harmonic maps from surfaces into the Minkowski sphere $\mathbb{S}^4_1 \subset \mathbb{R}^5_1$ and the conformal Gauss maps of Willmore surfaces in $\mathbb{S}^3$ (see [9]), and minimal surfaces in anti-de-Sitter space with its applications in the AdS/CFT correspondence in string theory (see [10,11]), we are aiming at extending the classical analytical methods developed for cases of compact and Riemannian targets to cases of more general targets, namely pseudo-Riemannian manifolds (which are likewise non-compact and non-Riemannian). Some attempts in this direction have been made in [12], where the author studied the regularity for weakly harmonic maps from surfaces into certain types of pseudo-Riemannian manifolds.

In this paper, we shall consider the compactness problem and study the blow-up behaviors for a sequence of harmonic maps from a closed Riemann surface into a standard stationary Lorentzian manifold—a type of model spacetimes arising in general relativity.

* Corresponding author.
E-mail addresses: xlhan@math.tsinghua.edu.cn (X. Han), liangzhao@bnu.edu.cn (L. Zhao), mizhu@sjtu.edu.cn (M. Zhu).

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To describe our problem, we need some notations first. A standard stationary Lorentzian manifold is a product manifold $\mathbb{R} \times M$ equipped with a Lorentzian metric of the following form:

$$g = -\beta(dt + \omega)^2 + g_M,$$

where $(\mathbb{R}, dt^2)$ is the 1-dimensional Euclidean space, (M, g_M) is a d-dimensional smooth compact Riemannian manifold which, by Nash's embedding theorem, is embedded isometrically into some Euclidean space $\mathbb{R}^n$, β is a positive C^∞ function on M and ω is a C^∞ 1-form on M. Although the Lorentzian manifold $\mathbb{R} \times M$ is not compact, to use some results for harmonic maps into compact Riemannian manifolds, we need M to be compact. For more details on such manifolds, we refer to [13,14].

Let (Σ, h) be a closed Riemann surface with a metric h. For a map $(t, u) \in C^\infty(\Sigma, \mathbb{R} \times M)$, we consider the following Lagrangian:

$$E_g(t, u) = \frac{1}{2}\int_\Sigma \left\{-\beta(u)|\nabla t + \omega_i(u)\nabla u^i|^2 + |\nabla u|^2\right\} dvol_h, \tag{1.1}$$

which is called the Lorentzian energy of the map (t, u) on Σ. It is easy to see that $E_g(\cdot, \cdot)$ is conformally invariant. A critical point (t, u) in $C^\infty(\Sigma, \mathbb{R}\times M)$ of the Lagrangian (1.1) is called a harmonic map from Σ into the standard stationary Lorentzian manifold $(\mathbb{R} \times M, g)$.

Via direct calculations, one can derive the following Euler–Lagrange equations (c. f. [12])

$$\begin{cases} -\mathrm{div}\{\beta(u)(\nabla t + \omega_i(u)\nabla u^i)\} = 0, \\ -\mathrm{div}\nabla u = \nu_l \nabla \nu_l \cdot \nabla u - H + \langle H, \nu_l\rangle \nu_l, \end{cases} \tag{1.2}$$

where $\{\nu_l, l = d+1, \ldots, n\}$ is an orthonormal frame along the map u for the normal bundle $T^\perp M$ in $\mathbb{R}^n$ and H is denoted by $H = (H^1, H^2, \ldots, H^n)$ with

$$H^j := \beta(\nabla t + \omega_i \nabla u^i)\cdot \nabla u^k \left(\frac{\partial \omega_j}{\partial y^k} - \frac{\partial \omega_k}{\partial y^j}\right) - \frac{1}{2}\frac{\partial \beta}{\partial y^j}|\nabla t + \omega_i \nabla u^i|^2.$$

Extending the Lagrangian $E_g(\cdot, \cdot)$ to maps in the space of $W^{1,2}$, one can naturally define a weakly harmonic map from Σ into $(\mathbb{R}\times M, g)$ as a critical point (t, u) in $W^{1,2}(\Sigma, \mathbb{R}\times M)$ of the Lagrangian $E_g(\cdot, \cdot)$. Compared with a classical harmonic map into a compact Riemannian manifold which has a potential being in L^2 and antisymmetric (see e.g. [2,3]), the system (1.2) considered here is a second order critical elliptic system with a potential being in L^2 but not necessary antisymmetric. By exploring the special structure of the system (1.2) and making use of the conservation law (due to the symmetry of the target generated by the timelike killing vector field ∂_t), one can adapt the methods developed by Rivière [2] and Rivière–Struwe [3] to get the regularity of weak solutions of (1.2). This is done in [12].

To study the compactness problem for such maps, we define E-energy of $(t, u) \in C^\infty(\Sigma, \mathbb{R} \times M)$ on a domain $U \subset \Sigma$ as follows:

$$E(t, u; U) := \int_U |\nabla t|^2 + |\nabla u|^2.$$

Then, our main result is the following:

Theorem 1.1. *Let $\{(t_k, u_k)\}$ be a sequence of smooth harmonic maps from Σ to $(\mathbb{R} \times M, g)$ with uniformly bounded E-energy:*

$$E(t_k, u_k) \le \Lambda < +\infty.$$

After taking a subsequence, still denoted by $\{(t_k, u_k)\}$, we can find a harmonic map $(t, u) : \Sigma \to (\mathbb{R} \times M, g)$ and a finite set $\mathcal{S} = \{p_1, p_2, \ldots, p_I\}$, such that, $\{(t_k, u_k)\}$ converges to (t, u) weakly in $W^{1,2}(\Sigma)$ and strongly in $W^{1,2}_{loc}(\Sigma \setminus \mathcal{S})$.

Furthermore, there is a finite set of harmonic spheres $(\sigma_i^l, \xi_i^l) : \mathbb{S}^2 \to (\mathbb{R} \times M, g)$, $i = 1, 2, \ldots, I$; $l = 1, 2, \ldots, L_i$, such that the following Lorentzian energy identity holds:

$$\lim_{k\to\infty} E_g(t_k, u_k) = E_g(t, u) + \sum_{i=1}^{I}\sum_{l=1}^{L_i} E_g(\sigma_i^l, \xi_i^l). \tag{1.3}$$

For E-energy, we have the following inequality:

$$\lim_{k\to\infty} E(t_k, u_k) \ge E(t, u) + \sum_{i=1}^{I}\sum_{l=1}^{L_i} E(\sigma_i^l, \xi_i^l). \tag{1.4}$$

Our proof of the above theorem follows the blow-up scheme by Ding–Tian in [15], which is based on the local singularity removability for a harmonic map from a punctured disk into a compact Riemannian manifold with bounded energy. However, in our case, Sacks–Uhlenbeck' method [7] cannot be applied to remove the local singularity, since the metric of the target manifold now is non-positive-definite. Fortunately, this difficulty can be overcome, thanks to the regularity result for weak solutions obtained in [12]. On the other hand, we remark that the holomorphicity property of the Hopf differential associated to a critical point of the Lagrangian E_g is preserved, which plays a key role in our blow-up analysis.