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Energy dependent integrability

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ABSTRACT

We review the conditions for separability of 2-dimensional natural Hamiltonian systems. We examine the possibility that the separability condition is satisfied on a given energy hypersurface only (weak integrability) and derive the additional requirement necessary to have separability at arbitrary values of the Hamiltonian (strong integrability). We give some new examples of systems admitting separating coordinates whose relation with the original ones explicitly depends on energy and provide a list of separable potentials discussing the nature of conserved quantities they admit.

1. Introduction

The study of integrability of Hamiltonian systems by means of geometric techniques naturally leads to the notions of *strong* and *weak* complete integrability [1–3]. In the case of weak integrability at fixed energy, the notion has its correspondence in the theory of variable separation for the *null* Hamilton–Jacobi equation [4]. Separability is associated with second-rank conformal Killing tensors and generates integrals of motion quadratic in the momenta [5]. Quadratic integrals at arbitrary and fixed energy (respectively, 'strongly' and 'weakly' conserved phase-space functions) can be treated in a unified way by exploiting the geometrization of the dynamics obtained by the introduction of the Jacobi metric. In this more general framework, nth degree polynomial integrals correspond to nth rank Killing tensors of a conformal Riemannian geometry. In this paper our main focus is to review the separability of 2-dimensional natural Hamiltonian systems. Then, only a single variational symmetry is needed.

Linear integrals are associated to Killing vector fields of a conformal geometry. They play a double role as Noether point symmetries and isometry generators [6]. Polynomial integrals of second or higher degree are associated to Killing tensors and correspond to Noether symmetries of the generalized[1] type [6,8]. Related approaches to the study of the structure of integrable and separable systems are those based on the study of Poisson algebras of integrals of motion [9], those connected with the bi-Hamiltonian nature of integrable systems [10] and the r-matrix and Lax-pair methods [11]. Moreover, it is worth mentioning approaches exploiting analytical properties of the solutions in an effort to explicitly integrating the equations of motion [12,13].

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[1] Here we are referring to symmetries having generators depending on time derivatives of the dependent variables in contrast to point symmetries [7].

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Here we would also like to recall the issue of the *identification* of systems possessing polynomial integrals of a given degree. A general procedure to find an integrability condition for the existence of a linear integral can be given a covariant form by means of the identification of a geometric invariant whose vanishing is equivalent to the validity of the integrability condition [14,15]. An analogous procedure has been used by Kruglikov [16] for higher-order integrals.

In the natural case, the integrability condition for quadratic integrals of motion involves an arbitrary analytic function $S(z)$ determining the *conformal factor* in the Jacobi Hamiltonian. For quadratic integrals at arbitrary energy, the function $S(z)$ is a second degree polynomial with real second derivative and the integrability condition then reduces to the classical condition of Darboux [17]. Thereafter, the possibility of searching for polynomial integrals of higher degree at arbitrary and/or fixed energy can also be addressed. Some examples of systems admitting a second quadratic integral at zero energy were provided in [18]. Here we present a new possibility that we can define as *energy-dependent separability* in which the conformal factor explicitly depends on energy, $S(z, E)$. Since the transformation to separating variables is given by rational powers of S, the separating coordinates depend themselves on energy as an independent parameter. This dependence is henceforth induced into the Killing tensor invariant, giving an effective higher-order integral. We provide explicit examples in the case of fourth-order invariants at arbitrary energy corresponding to second-rank Killing tensors.

The structure of the paper is the following: in Section 2 we recall the approach to integrability and separability of 2 dimensional systems based on conformal transformations with a detailed discussion of Killing tensor equations; in Section 3 the case of second-rank Killing tensor is detailed in the weak and strong cases; in Section 4, some examples of linear and quadratic weak invariants are recalled; in Section 5 is introduced a class of fourth order invariants at arbitrary energy corresponding to second rank Killing tensors and in Section 6 there are some concluding remarks with a particular emphasis on systems with indefinite metrics.

2. Separability at fixed and arbitrary energy

As suggested by Hietarinta (see [19], section 7.2), the form of a *null* Hamiltonian is preserved under canonical point transformations generated by analytical functions. We will show in this section that for 2-dimensional systems such transformations are in fact conformal transformations related to time reparametrization of the dynamics.

2.1. Null Hamiltonian and conformal coordinate transformations

In general the Hamiltonian itself has the form $H = T + V = E$, where T is a quadratic form in the momenta and V is a function on the configuration space. The possibility of linear terms in the momenta greatly complicates things [20] so that for simplicity we limit the discussion to time-reversible systems. For any given energy E of the system, to represent the dynamics, we can use the *null* Hamiltonian

$$\mathcal{H}(p, q; E) \doteq H - E = T - G, \quad G(q; E) \doteq E - V(q), \tag{1}$$

provided that we impose the constraint $\mathcal{H} = 0$. For any such zero energy Hamiltonian we can reparametrize the system by introducing a new time variable $\bar{t}$ defined by the relation

$$d\bar{t} = \Lambda(p, q)dt, \tag{2}$$

together with a redefined Hamiltonian

$$\bar{\mathcal{H}} = \mathcal{H}/\Lambda(p, q) = T/\Lambda - G/\Lambda = 0. \tag{3}$$

The new Hamiltonian will then give the same equations of motion on the constraint surface $\bar{\mathcal{H}} = 0$. The function $\Lambda(p, q)$ corresponds to the inverse of the *lapse function* as used in cosmological applications. The lapse determines the independent variable gauge and gives the rate of physical time change relative to coordinate time. The lapse function can be taken as any non-zero function on the phase space. For a Riemannian metric

$$T = G = \frac{1}{2} h^{\alpha\beta} p_\alpha p_\beta, \tag{4}$$

we can use the lapse function and the Jacobi trick to include in the treatment also motion on curved surfaces. By the lapse choice

$$\Lambda := G = E - V, \tag{5}$$

the resulting Hamiltonian $\bar{\mathcal{H}}$ gives us the 'Jacobi Hamiltonian'

$$H_J := [2(E - V)]^{-1} h^{\alpha\beta} p_\alpha p_\beta, \tag{6}$$

which is the purely kinetic Hamiltonian for the geodesics of the Riemannian geometry with metric

$$g_{\alpha\beta} := G h_{\alpha\beta}, \tag{7}$$

which is manifestly conformally related to $h_{\alpha\beta}$ by the factor $G(q)$.