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Classical and quantum monodromy via action–angle variables

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ABSTRACT

We give an elementary description of the relationship between the classical and quantum monodromy of a completely integrable system, from the point of view of geometric quantization, as a consequence of the construction of action–angle variables. We also describe the relation to Symington's notion of "affine monodromy".

1. Introduction

The celebrated Arnol'd–Liouville Theorem gives the structure of a completely integrable system near a regular level in terms of *action–angle coordinates.* Duistermaat addressed the question of the existence of *global* action–angle coordinates in his seminal paper [1]. One obstruction to the existence of global action–angle coordinates is the *monodromy*, which measures the failure of the torus bundle defined by the Liouville tori to be trivial.

There turns out to be a manifestation of monodromy in *quantum* systems as well. This was first observed by Cushman and Duistermaat for the spherical pendulum in [2], and since then "quantum monodromy" has become an active area of research, among physicists and molecular chemists as well as mathematicians. The relationship between the classical and quantum monodromy of a system was proved rigorously in the context of pseudodifferential quantization by Vu Ngoc in [3]. He explains:

> ...when using appropriate tools, the link between classical and quantum monodromy becomes more or less trivial. Of course, there is a price to pay: these tools (Fourier Integral Operators, to name one of the most important) are actually quite delicate to define.

The purpose of this note is to show that these delicate tools are not necessary to show the relationship between the classical and quantum monodromy, and that the link can be easily explained in the context of geometric quantization.

This description is not really new; essentially the same argument is used by Sadovskii, Zhilinskii, and collaborators in the context of EBK quantization, for example in [4]. Similar ideas do underlie some of Vu Ngoc's arguments, for example in [3] and [5], but they can be difficult to isolate, and the essential ideas are so simple that I thought it worth laying them

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out clearly in one place. I have also not seen them applied specifically to geometric quantization, and I thought it interesting that the connection between classical and quantum monodromy becomes so straightforward from this perspective. The argument is valid for any quantization scheme with a Bohr–Sommerfeld condition given by integer action variables, as in Section 3.1.

(Sansonetto and Spera also discuss monodromy in the framework of geometric quantization in [6], although their approach is somewhat different: They relate the classical monodromy to a choice of prequantum connection, and do not explicitly mention quantum monodromy, although there are references to the effect of the monodromy on quantum operators. Also, recently, in [7] Cushman and Śniatycki have taken a different approach to extending geometric quantization to systems with monodromy, as part of their program of "Bohr–Sommerfeld–Heisenberg quantization" as described in [8].)

Symington in [9] further clarifies the place of monodromy in the structure of the phase spaces, pointing out that "any topological monodromy of a regular Lagrangian fibration is reflected in the global geometry of the base". ([9], §2.3) She defines the concept of *affine monodromy* as distinct from "topological monodromy", and shows that the affine monodromy matrix is simply the inverse transpose of the topological monodromy matrix. We show that the affine monodromy is in fact the derivative of our formulation of quantum monodromy, giving an interpretation of the quantum monodromy as a reflection of the structure of the classical phase space.

The argument can be summarized in two paragraphs. The (classical) monodromy describes how a set of cycles forming a homology basis for H^1 of the fibre changes as we go around a loop in the base. Action coordinates can be computed by integrating the Liouville 1-form over exactly such a set of cycles, so the monodromy describes how the (locally defined) action coordinates change as we go around the loop.

In local action–angle coordinates near any regular leaf, the Bohr–Sommerfeld fibres are those all of whose action coordinates are integers. Even if the action–angle coordinates are only locally defined, the condition of "action coordinates are integers" makes sense globally. The "new" and "old" action coordinates give two bases of the local lattice of quantum states, the relation between which is the quantum monodromy. Since the action coordinates are related by the classical monodromy, the quantum monodromy is the same as the classical, modulo a transpose coming from the difference between vectors and their coordinate representation, and an inverse coming from solving a system of equations.

2. Integrable systems

Let (X, ω) be a symplectic manifold of dimension $2n$. A *completely integrable system* is a collection of n functions $f_1, \ldots, f_n$ which pairwise Poisson commute, and which are independent almost everywhere. Let $F = (f_1, \ldots, f_n): X \to \mathbb{R}^n$, and let $B = F(X)$. For $c \in B$, let F_c denote the c-level set of F, which we will generally assume to be compact and connected.

A canonical example of a completely integrable system is the energy–momentum map for the spherical pendulum, where $f_1 = E$ is the energy and $f_2 = L$ is the angular momentum. There are of course many other examples (a long list is given in §1.4 of [10]), and there is an extensive literature on integrable systems.

2.1. Arnol'd–Liouville and action–angle variables

The local structure of an integrable system is described by the Arnol'd–Liouville theorem, which gives a description of the system in terms of particularly simple coordinates called *action–angle coordinates.*

Theorem 1 (*Arnol'd–Liouville*)**.** *Let $c \in B$ be a regular value of an integrable system $F = (f_1, \ldots, f_n): X \to B$ and let $F_c = F^{-1}(c)$. Then F_c is a Lagrangian submanifold of X.*

Furthermore, assume F_c is compact and connected. Then there is a neighbourhood U of F_c in X and a diffeomorphism $(a, \alpha): U \to V \times T^n$, where V is an open subset of $\mathbb{R}^n$ and $T^n = (S^1)^n$ is a torus, such that (a, α) are symplectic coordinates, and F is a function of a only.

The coordinates (a, α) are called action–angle coordinates.

We can describe the conclusions of the theorem as follows (as in [11]):

1. The fibre F_c is diffeomorphic to a torus T^n, on which there are coordinates $\alpha_1, \ldots, \alpha_n$ in which the flow of the Hamiltonian vector fields of $f_1, \ldots, f_n$ are linear.
2. There is a complementary set of coordinates $a_1, \ldots, a_n$, called *action coordinates,* which Poisson commute with all f_j, such that the (a, α) form a symplectic chart.

See [11], Thm 18.12; [1], Thm 1.1; or [12], §§49–50. (Also discussed at length in [13], section II.2.) The above phrasing follows [1].

Action coordinates can be computed as follows.

Theorem 2 (*[1], Thm 1.2*)**.** *Let $c \in B$ be a regular value of F. Choose a neighbourhood $V \subset B$ of c consisting of regular values such that ω is exact on $F^{-1}(V)$, and let Θ be a primitive for ω. (Such a Θ exists by, for example, the Weinstein Lagrangian*