ARTICLE IN PRESS

Journal of Geometry and Physics ▮ (▮▮▮▮) ▮▮▮–▮▮▮

Contents lists available at ScienceDirect

Journal of Geometry and Physics

journal homepage: www.elsevier.com/locate/jgp

Integrability via reversibility

Maciej P. Wojtkowski

Department of Mathematics Physics and Computer Science, University of Opole, 45-052 Opole, Poland

ARTICLE INFO

Article history:
Received 18 December 2015
Received in revised form 6 May 2016
Accepted 23 July 2016
Available online xxxx

Keywords:
Integrability
Reversibility
Coexistence of integrability and chaotic behavior

ABSTRACT

A class of left-invariant second order reversible systems with functional parameter is introduced which exhibits the phenomenon of robust integrability: an open and dense subset of the phase space is filled with invariant tori carrying quasi-periodic motions, and this behavior persists under perturbations within the class.

Real-analytic volume preserving systems are found in this class which have positive Lyapunov exponents on an open subset, and the complement filled with invariant tori.

1. Introduction

We study a family of second order dynamical systems on a locally homogeneous Riemannian space M, modeled on a special solvable Lie group. The simplest example is the geodesic flow of a left-invariant metric. Our class generalizes the examples discovered by Butler [1], and Bolsinov and Taimanov [2]. In these examples the complete integrability of the geodesic flow in the tangent bundle TM is accompanied by highly non-integrable behavior on an invariant submanifold of codimension $n = dim\ M$. The dynamics there is the suspension of a toral automorphism, and in [2] the hyperbolic automorphism is chosen, which leads to an Anosov flow. The presence of an Anosov flow as a subsystem guarantees the positivity of topological entropy.

We show that such a behavior extends to a larger class of left-invariant second order systems. This class is parametrized by a matrix L and a functional parameter F: a smooth vector field in the unit ball of the Euclidean space. We call them $L - F$ *systems*. The equations of an $L - F$ system can be written in coordinates $(w, u; \xi, \eta)$ where $w \in \mathbb{R}^n$, $\xi \in \mathbb{R}^n$, and u and η are real variables

$$\begin{aligned} \frac{d\xi}{dt} &= \eta F(\xi), \qquad \frac{d\eta}{dt} = -\langle F(\xi), \xi\rangle \\ \frac{dw}{dt} &= e^{uL}\xi, \qquad \frac{du}{dt} = \eta. \end{aligned} \tag{1}$$

The equations in the first line form a factor of the whole system, since they involve only the variables (ξ, η). We call them the *Euler equations*, since they come from the usual projection of a left invariant system from the tangent bundle of a Lie group to the Lie algebra. The Euler equations preserve spheres $\{\xi^2 + \eta^2 = const\}$, and we will restrict the Euler system to the unit sphere. Further for a given rank n lattice $\Gamma_0 \subset \mathbb{R}^n$ we can consider the variables w modulo Γ_0 so that the configuration space of our system becomes the manifold $M = \mathbb{T}^n \times \mathbb{R} \ni (w, u)$, where the torus $\mathbb{T}^n = \mathbb{R}^n/\Gamma_0$.

To obtain a compact configuration space we need to assume that the linear map $A = e^L$ is an automorphism of the lattice Γ_0. In this case we can glue the tori $\{u = 0\}$ and $\{u = 1\}$ by the automorphism A. Namely we identify the points $(w, 0)$ and

E-mail address: mpwojtkowski@math.uni.opole.pl.

http://dx.doi.org/10.1016/j.geomphys.2016.07.015
0393-0440/

Please cite this article in press as: M.P. Wojtkowski, Integrability via reversibility, Journal of Geometry and Physics (2016), http://dx.doi.org/10.1016/j.geomphys.2016.07.015

$(Aw, 1)$ for every w. The compact configuration space will be denoted by N. In our case it is a solvmanifold, a left quotient $N = \Gamma \setminus G$ of a solvable Lie group G by a lattice $\Gamma \supset \Gamma_0$.

The crucial property of $L - F$ systems is their J-reversibility, where J is the involution of the tangent bundle TM given by $J(w, u; \xi, \eta) = (-w, u; \xi, -\eta)$. Let us recall that a system is *J-reversible* if the involution J conjugates the forward in time dynamics with the backward in time dynamics.

There is a vast literature devoted to J-reversible systems. The survey paper of Lamb and Roberts [3] contains an extensive bibliography.

In particular there is a version of the KAM theory for the J-reversible systems. It goes back to Moser [4], and Sevryuk [5]. In our case we establish robust integrability: it persists under any small perturbation as long as we stay in the family of $L - F$ systems. Note that this family is parametrized by an infinite dimensional Banach space of vector fields F. What is notable is that we do not assume volume preservation, the symmetries imposed on the system force the integrability, and the occurrence of a finite absolutely continuous invariant measure. This measure has a density with respect to the Liouville volume which is only C^∞, and typically no real-analytic invariant density exists (Section 8).

The J-reversible KAM theory would give us large subsets of quasi-periodic motions for perturbations which are not left-invariant, as long as they are J-reversible. We were unable to check the non-degeneracy of the unperturbed system required for the application of the KAM theory. However we conjecture that the non-degeneracy does hold for most systems under consideration.

Butler, [6,7], used the mechanism discovered in [2] to obtain C^∞ examples of integrable volume preserving systems with positive metric entropy. In our class we find whole families of real-analytic systems with positive metric entropy and a subset filled with quasi-periodic motions, open but not dense (Section 9).

The phenomenon of robust integrability, accompanied by positive topological entropy occurs in particular for geodesic flows of linear connections. The generalization of the geodesic flow of the Levi-Civita connection to more general linear connections was discussed in [8]. Such a generalization appears naturally in the study of Gaussian thermostats, a class of systems introduced by Hoover [9]. The paper of Gallavotti and Ruelle [10] introduces the Gaussian thermostats in the physical context. In particular in our class of systems we find a Gaussian thermostat with the following paradoxical behavior (Section 10). For small kinetic energy the system is asymptotic to an Anosov flow, which has a large codimension in the whole phase space. For larger values of the kinetic energy the system undergoes a drastic change, it becomes integrable: an open and dense subset in the phase space is filled with quasi-periodic motions. The Anosov subsystem is still present, but for a subset of initial conditions of full Lebesgue measure the solutions stay away from that chaotic subsystem, and fill densely invariant tori.

The author would like to thank Gabriela Tereszkiewicz, Wojciech Czernous and Adam Doliwa for very useful discussions. Preliminary results of this work were presented at the seminar of Lai-Sang Young at the Courant Institute in November 2014. We are grateful for her insights and hospitality.

Finally we thank the anonymous reviewer for very useful suggestions to improve the text.

2. The configuration space

Our configuration space is a locally homogeneous Riemannian space modeled on a special Lie group G. We start its description with the Lie algebra $\mathfrak{g}$. We assume that *dim* $\mathfrak{g} = n + 1$ and that $\mathfrak{g}$ contains an n dimensional abelian ideal $\mathfrak{g}_0$. We choose an arbitrary scalar product in $\mathfrak{g}$, and let b denote a unit vector orthogonal to $\mathfrak{g}_0$. Since the ideal $\mathfrak{g}_0$ is assumed to be abelian the Jacobi identity imposes no conditions on the operator $L : \mathfrak{g}_0 \to \mathfrak{g}_0, L = ad_b$. At this stage we place no restrictions on the operator L. Later on we will consider various special cases. We endow the Lie group with the left invariant metric determined by our choice of the scalar product in $\mathfrak{g}$.

The Levi-Civita connection ∇ on the Riemannian manifold G can be expressed as a tensor on $\mathfrak{g}$. It can be calculated directly, which is done in the fundamental paper of Milnor [11], where extensive explanations can be found. The formulas read

$$\begin{aligned} &\nabla_b b = 0, \qquad \nabla_b \xi = A\xi, \quad \text{for } \xi \in \mathfrak{g}_0, \\ &\nabla_\xi b = -S\xi, \qquad \nabla_\xi \zeta = \langle S\xi, \zeta\rangle b, \quad \text{for } \xi, \zeta \in \mathfrak{g}_0 \end{aligned} \tag{2}$$

where $S = \frac{1}{2}(L + L^*)$ and $A = \frac{1}{2}(L - L^*)$ denote the symmetric and skew-symmetric parts of the operator L.

The Lie algebra has an important automorphism $K : \mathfrak{g} \to \mathfrak{g}$, where $-K$ is equal to the euclidean reflection in $\mathfrak{g}_0$. There are only few Lie algebras with this kind of additional symmetry. It is not difficult to enumerate all of them. Our class of Lie algebras is singled out by the additional property that the invariant subspace of K is a subalgebra.

Since the automorphism K is orthogonal it generates the automorphism $\mathcal{K}$ of the Lie group G which is an isometry. This isometry will play crucial role in our discussion. Let us note that both K and $\mathcal{K}$ are involutive, i.e., $K = K^{-1}$, $\mathcal{K} = \mathcal{K}^{-1}$.

The Lie group has the following matrix representation, which we will also denote by G. It consists of matrices with the block form

$$\begin{bmatrix} 1 & 0 \\ w & e^{uL} \end{bmatrix}, \quad w \in \mathbb{R}^n, u \in \mathbb{R}.$$

Please cite this article in press as: M.P. Wojtkowski, Integrability via reversibility, Journal of Geometry and Physics (2016), http://dx.doi.org/10.1016/j.geomphys.2016.07.015