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Monodromy of Hamiltonian systems with complexity 1 torus actions

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ABSTRACT

We consider the monodromy of n-torus bundles in n degree of freedom integrable Hamiltonian systems with a complexity 1 torus action, that is, a Hamiltonian $\mathbb{T}^{n-1}$ action. We show that orbits with $\mathbb{T}^1$ isotropy are associated to non-trivial monodromy and we give a simple formula for computing the monodromy matrix in this case. In the case of 2 degree of freedom systems such orbits correspond to fixed points of the $\mathbb{T}^1$ action. Thus we demonstrate that, given a $\mathbb{T}^{n-1}$ invariant Hamiltonian H, it is the $\mathbb{T}^{n-1}$ action, rather than H, that determines monodromy.

1. Introduction

The, now classical, work by Duistermaat on obstructions to global action–angle coordinates in integrable Hamiltonian systems [1] highlighted the importance of the non-triviality of torus bundles over circles for such systems. Since then non-trivial monodromy has been demonstrated in several integrable Hamiltonian systems. We indicatively mention the spherical pendulum [1,2], the Lagrange top [3], the Hamiltonian Hopf bifurcation [4], the champagne bottle [5], the coupled angular momenta [6], the two-centers problem [7], and the quadratic spherical pendulum [8,9]. A common aspect of these systems is the presence of a symmetry given by a Hamiltonian $\mathbb{T}^{n-k}$ action, where n is the number of degrees of freedom (for the two-centers problem $k = 2$ and for the other systems $k = 1$).

Remark 1.1. Hamiltonian $\mathbb{T}^{n-k}$ actions on symplectic $2n$ manifolds are called *complexity k torus actions*. Classification of symplectic manifolds with such actions has been studied by Delzant in [10] ($k = 0$), and Karshon and Tolman in [11] ($k = 1$). We note that for integrable systems with a complexity 0 torus action monodromy is always trivial.

In the present paper we consider integrable n degree of freedom systems with a complexity 1 torus action, that is, a Hamiltonian $\mathbb{T}^{n-1}$ action. Monodromy in such systems (along a given curve) is determined by $n-1$ free integer parameters. We will show that these parameters are related to singular orbits of the $\mathbb{T}^{n-1}$ action via the *curvature form* of an appropriate principal $\mathbb{T}^{n-1}$ bundle; see Theorems 3.2 and 3.4. Surprisingly, this relation has not been observed before. The usually adopted approaches to monodromy (see [12–17,2]) do not take into account the differential geometric invariants of the Hamiltonian $\mathbb{T}^{n-1}$ symmetry, such as the curvature form and the *Chern numbers*. Moreover, these approaches are rather concentrated on the study of the whole *integral map*, that is, the Hamiltonian and the momenta that generate the $\mathbb{T}^{n-1}$ action. Our results in this paper show that the Hamiltonian plays a secondary role to the momenta for determining monodromy.

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Specifically, for 2 degrees of freedom systems monodromy is determined in terms of the fixed points of the Hamiltonian $\mathbb{T}^1$ action; see Theorem 3.10. For n degree of freedom systems monodromy is determined in terms of how the $\mathbb{S}^1$ isotropy group is expressed in a given basis of the $\mathbb{T}^{n-1}$ action; see Theorem 3.15.

The paper is organized as follows. In Section 2 we specify our setting and recall necessary definitions from the theory of principal bundles. In Section 3 we formulate our main results that relate monodromy with singularities of the $\mathbb{T}^{n-1}$ action; see Theorems 3.2, 3.4, 3.10 and 3.15. The proof of Theorem 3.2, which is more technical, is postponed to Section 5. In Section 4 we apply our techniques to various integrable systems. The paper is concluded in Section 6 with a discussion.

2. Preliminaries

Let M be a connected $2n$-dimensional manifold with a symplectic form Ω. Since Ω is a non-degenerate 2-form, to every smooth function $F_1 \colon M \to \mathbb{R}$ one can associate the so-called *Hamiltonian vector field* $X_{F_1} = \Omega^{-1}(dF_1)$. Suppose that we have n almost everywhere independent functions $F_1, \dots, F_n$ on M such that all *Poisson brackets* vanish:

$$\{F_i, F_j\} = \Omega(X_{F_i}, X_{F_j}) = 0.$$

Then we say that we have an integrable Hamiltonian system on M. The map

$$(F_1, \dots, F_n) \colon M \to \mathbb{R}^n$$

is called the *integral map* of the system. Everywhere in the paper we assume that the Assumption 2.1 hold (except for Section 5 where we work in a more general setting of a Hamiltonian $\mathbb{T}^k$ action, $1 \le k \le n-1$).

Assumptions 2.1. The integral map F is assumed to have the following properties.

(1) F is *proper*, that is, for every compact set $K \subset \mathbb{R}^n$ the preimage $F^{-1}(K)$ is a compact subset of M.
(2) The integral map F is invariant under a Hamiltonian $\mathbb{T}^{n-1}$ action.
(3) The $\mathbb{T}^{n-1}$ action is free on $F^{-1}(R)$, where $R \subset \text{image}(F)$ the set of regular values of F.

Consider a regular simple closed curve $\gamma \subset R$ and assume that the fibers $F^{-1}(\xi),\ \xi \in \gamma$, are connected. By the Arnol'd–Liouville theorem we have a n-torus bundle

$$(E_\gamma = F^{-1}(\gamma), \gamma, F) \tag{1}$$

with respect to F. Take a fiber $F^{-1}(\xi_0),\ \xi_0 \in \gamma$, and let T^{n-1} be any orbit of the Hamiltonian $\mathbb{T}^{n-1}$ action on $F^{-1}(\xi_0)$. We choose a basis $(e_1, \dots, e_n)$ of the integer homology group $H_1(F^{-1}(\xi_0))$ so that $(e_1, \dots, e_{n-1})$ is a basis of $H_1(T^{n-1})$. Since the Hamiltonian $\mathbb{T}^{n-1}$ action is globally defined on E_γ, the generators $e_j,\ j = 1, \dots, n-1$, are also 'globally defined', that is they are preserved under the parallel transport along γ. It follows that the monodromy matrix of the bundle (E_γ, γ, F) with respect to the basis $(e_1, \dots, e_n)$ has the form

$$\begin{pmatrix} 1 & \cdots & 0 & m_1 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & m_{n-1} \\ 0 & \cdots & 0 & 1 \end{pmatrix}.$$

We call $\vec{m} = (m_1, \dots, m_{n-1}) \in \mathbb{Z}^{n-1}$ the *monodromy vector*. In Section 3 we relate $\vec{m}$ to the curvature form of an appropriate principal $\mathbb{T}^{n-1}$ bundle and then give a formula that allows us to compute $\vec{m}$ in specific integrable Hamiltonian systems.

The assumption of the existence of a $\mathbb{T}^{n-1}$ action made throughout this paper brings us in the context of principal torus bundles and their Chern numbers. We recall here some relevant definitions. For a detailed exposition of the theory we refer to Postnikov [18].

Consider a principal $\mathbb{T}^{n-1}$ bundle (E, B, ρ). The structure group $\mathbb{T}^{n-1}$ is isomorphic to the direct product of $n-1$ circles:

$$\mathbb{T}^{n-1} = \{(e^{i\varphi_1}, \dots, e^{i\varphi_{n-1}}) \mid \varphi_j \in \mathbb{R}\} \subset \mathbb{C}^{n-1}.$$

The Lie algebra $T_e\mathbb{T}^{n-1}$ can be thus identified with $i\mathbb{R}^{n-1}$. The Lie bracket is identically zero since $\mathbb{T}^{n-1}$ is a commutative group.

Let $A^\#$ denote the fundamental vector field generated by $A \in i\mathbb{R}^{n-1}$ and $R_g^\star$ denote the pull-back of the right shift $R_g \colon E \to E$.

Definition 2.2. A *connection one-form* ω *on* (E, B, ρ) is a $i\mathbb{R}^{n-1}$-valued one-form on E such that $\omega(A^\#) = A$ and $R_g^\star(\omega) = \omega$.

Remark 2.3. In our setting both E and B are compact manifolds. Hence a connection one-form exists. It separates tangent spaces of E into *vertical* and *horizontal* subspaces.

Let $\{U_\alpha\}_{\alpha \in I}$ be a trivialization cover of B.

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