ARTICLE IN PRESS
Journal of Geometry and Physics ▮ (▮▮▮▮) ▮▮▮–▮▮▮
Contents lists available at ScienceDirect
Journal of Geometry and Physics
journal homepage: www.elsevier.com/locate/jgp

Weakly Hamiltonian actions

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ARTICLE INFO

Article history:
Received 3 February 2016
Received in revised form 27 April 2016
Accepted 28 April 2016
Available online xxxx

Keywords:
Weakly Hamiltonian actions
Integrable systems
Lie algebras
Poisson geometry
Real analytic functions

ABSTRACT

In this paper we generalize constructions of non-commutative integrable systems to the context of weakly Hamiltonian actions on Poisson manifolds. In particular we prove that abelian weakly Hamiltonian actions on symplectic manifolds split into Hamiltonian and non-Hamiltonian factors, and explore generalizations in the Poisson setting.

1. Introduction

An integrable system on a $2n$-dimensional symplectic manifold is given by n generically independent pairwise commuting functions. More generally, a non-commutative integrable system is determined by a set of $2n - r$ integrals ($r \leq n$), out of which r do pairwise commute. Integrable systems come with *infinitesimal* abelian actions which are Hamiltonian, in the sense that they have an equivariant *momentum map*.

Furthermore, under compactness assumption on the invariant sets these infinitesimal abelian actions integrate into a torus action for which there is a normal form (action–angle coordinates).

However, some discrete integrable systems [1] do not present commuting first integrals but rather commuting flows. Moreover, there are systems that become Hamiltonian after reduction by non-commutative symmetries. This justifies considering a more general framework where *weakly Hamiltonian actions* take over Hamiltonian actions. In this paper we look at (infinitesimal) actions of abelian Lie algebras on Poisson manifolds having first integrals, but that cannot be arranged into an equivariant momentum map. Our main purpose is discussing conditions under which one can still find "invariant subsets" (Poisson submanifolds) where the residual action is indeed Hamiltonian.

In the symplectic setting there is already a result of Souriau in this direction [2, Theorem 13.15], which gives a conceptual explanation of a classical theorem of König on the decomposition of both the kinetic energy and the motion of a system. However, this is a result for actions of arbitrary Lie groups (the Galilean group in König's theorem) and our main point here is to stress that in the abelian case one can go further and obtain a splitting not just of the symplectic manifold, but also of the action into a weakly Hamiltonian and a Hamiltonian factor. Moreover, in the much richer Poisson setting we will see how mild conditions on the action determine Poisson submanifolds, and how under stronger hypotheses a splitting for the Poisson structure can be found, and in some cases, even for the action. In fact, our global construction – when specialized to

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http://dx.doi.org/10.1016/j.geomphys.2016.04.022
0393-0440/

Please cite this article in press as: D. Martínez Torres, E. Miranda, Weakly Hamiltonian actions, Journal of Geometry and Physics (2016), http://dx.doi.org/10.1016/j.geomphys.2016.04.022

a local setting – has strong reminiscences of the classical Weinstein local splitting theorem in Poisson geometry [3], and we expect to be able to relate other local splitting results with our construction. Finally our techniques – which are different from Souriau's – are also appropriate to draw global conclusions for actions of non-abelian Lie algebras.

2. Motivating examples

We start by presenting three different types of (complete) weakly Hamiltonian abelian actions in symplectic and Poisson manifolds, and we close the section with a related example.

2.1. Standard action by translations

The paradigm of abelian symmetries by Hamiltonian vector fields, but not fitting into a Hamiltonian action, is that of a symplectic vector space (V, σ) on which V itself – seen as an abelian Lie group – acts by translations: to each vector $v \in V$ we can assign a first integral which is the unique linear function $H_u \in V^*$ such that $dH_u = i_u\sigma$ (or rather, an affine one with that linear part). Since we have:

$$\{H_u, H_v\} = \sigma(u, v)$$

we will never be able to find a basis of first integrals in involution. Notwithstanding, any such choice of first integrals yields a weakly Hamiltonian action (indeed a non-commutative system with constant brackets).

2.2. The Galilean group

Let $G(3)$ be the Galilean group and consider its standard representation (cf. (13.7) in [2]) on $T^*\mathbb{R}^3$ with position and momentum coordinates q_i, p_i and particle mass m. Recall that $G(3)$ is an extension of the Euclidean group $E(3)$, and the restriction of this representation to $E(3)$ is Hamiltonian because it is the cotangent lift of its defining action on $\mathbb{R}^3$. However the Hamiltonian functions corresponding to the Galilean boosts mq_i and translations in the same direction p_i do not commute; indeed their Poisson bracket is the mass, and the corresponding cocycle is not exact, bringing in another example of weakly Hamiltonian action.

2.3. Weakly Hamiltonian actions and nilpotent Lie algebras

The symplectic form on the symplectic vector space (V, σ) can be interpreted as a 2-cocycle, and as such it gives rise to $\mathfrak{g} = \mathbb{R} \oplus_\omega V$ a central extension of the abelian algebra V. This Heisenberg type-Lie algebra is nilpotent with one dimensional center. The coadjoint orbit corresponding to the affine hyperplane:

$$\{\alpha \in \mathfrak{g}^* \mid \alpha(1, 0) = 1\}$$

can be canonically identified with V, and the restriction of the coadjoint action to this orbit is the linear action by translations above (2.1).

More generally, let $\mathfrak{g}$ be a nilpotent Lie algebra such that $[\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{z}(\mathfrak{g})$ (a 2-step nilpotent Lie algebra). As brackets lie in the center, they become Casimirs as functions on $\mathfrak{g}^*$, and therefore constants on coadjoint orbits. Then any subspace of $\mathfrak{g}$ intersecting trivially with the center provides an abelian Lie algebra acting in a weakly Hamiltonian fashion (which is not Hamiltonian provided that some of the brackets are non-trivial) on any coadjoint orbit (and in fact on the whole $\mathfrak{g}^*$). We illustrate this with the following low dimensional example (additional ones can be found by inspecting the list of low dimensional nilpotent Lie algebras up to dimension 7 [4–6]).

The nilpotent Lie algebra of dimension 6, $A^a_{6,5}$ (for $a \neq 0$) for which the non-vanishing relations on a base (see table III in [4]) are $[e_1, e_3] = e_5$, $[e_1, e_4] = e_6$, $[e_2, e_3] = ae_6$, $[e_2, e_4] = e_5$. In this case, the symplectic foliation by coadjoint orbits is given by e_6^* and e_5^*, thus defining a foliation with regular 4-dimensional symplectic leaves away from zero. The subspace spanned by e_1, e_2, e_3 and e_4 acts by commuting Hamiltonian vector fields but without momentum map, providing an example of weakly Hamiltonian action on a Poisson manifold.

2.4. Related examples

In [1], motivated by the study of discrete integrable systems, the "multi-time" Legendre transform is applied to multi-time Euler–Lagrange equations to obtain a system of commuting Hamiltonian flows. As observed in [1] this situation corresponds to having functions with constant Poisson brackets but these brackets are not necessarily zero,[1] thus providing an extra motivation to consider weakly Hamiltonian actions.

[1] As observed by the author in [1] indeed the vanishing of these Poisson brackets is equivalent to Lagrangian 1-form employed in the construction being closed on the solutions of the Euler–Lagrange equations.