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Billiard transformations of parallel flows: A periscope theorem

Alexander Plakhov [a,b], Serge Tabachnikov [c,*], Dmitry Treschev [d]

[a] Center for R&D in Mathematics and Applications, Department of Mathematics, University of Aveiro, Portugal

[b] Institute for Information Transmission Problems, Moscow, Russia

[c] Department of Mathematics, Penn State University, United States

[d] Steklov Mathematical Institute of Russian Academy of Science and Lomonosov Moscow State University, Russia

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ABSTRACT

We consider the following problem: given two parallel and identically oriented bundles of light rays in $\mathbb{R}^{n+1}$ and given a diffeomorphism between the rays of the former bundle and the rays of the latter one, is it possible to realize this diffeomorphism by means of several mirror reflections? We prove that a 2-mirror realization is possible if and only if the diffeomorphism is the gradient of a function. We further prove that any orientation reversing diffeomorphism of domains in $\mathbb{R}^2$ is locally the composition of two gradient diffeomorphisms, and therefore can be realized by 4 mirror reflections of light rays in $\mathbb{R}^3$, while an orientation preserving diffeomorphism can be realized by 6 reflections. In general, we prove that an (orientation reversing or preserving) diffeomorphism of wave fronts of two normal families of light rays in $\mathbb{R}^3$ can be realized by 6 or 7 reflections.

1. Introduction

This paper concerns geometrical optics, a classical subject that goes all the way back to Fermat, Huygens, Newton, and that remains a research area of a considerable contemporary interest.

One of the reasons for this interest is that, fairly recently, industrial methods were developed for manufacturing freeform reflective surfaces of optical quality.

From the mathematical point of view, a freeform mirror is a smooth hypersurface in Euclidean space from which the rays of light reflect according to the familiar law "the angle of incidence equals the angle of reflection". We refer to [1] for a historical introduction to geometrical optic and to [2] for an encyclopedic study of the subject. We cannot help mentioning a classical source, the treatise by W.R. Hamilton [3].

Another reason for the popularity of geometrical optics is its close relation with the ever-growing study of mathematical billiards; the reader interested in billiards is referred to [4–6]. Mathematical billiards describe the motion of a mass-point inside a domain with the elastic reflection off the boundary given by the law of equal angles.

Still another reason for which geometrical optics continues to attract attention is that the space of oriented lines, i.e., rays of light, in $\mathbb{R}^{n+1}$ is an example of a symplectic manifold (symplectomorphic to the cotangent bundle T^*S^n). The optical, or billiard, reflection in a mirror defines a symplectic transformation of the space of lines; see [7,6] for a modern treatment.

* Corresponding author.
E-mail address: tabachni@math.psu.edu (S. Tabachnikov).

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Only a negligible part of symplectic transformations of the space of lines is realized by the composition of reflections in mirrors. Indeed, a symplectic transformation of the space of rays in $\mathbb{R}^{n+1}$ is given by its generating function, a function of $2n$ variables, whereas a mirror is locally the graph of a function of only n variables. It is an interesting and, to the best of our knowledge, completely open problem to characterize the symplectic transformations of the space of oriented lines that arise as consecutive mirror reflections.

A common object of study in geometrical optics is a normal family of lines in $\mathbb{R}^{n+1}$, that is, an n-parameter family of oriented lines perpendicular to a hypersurface (think of this surface as emanating light). The hypersurface is called a wave front. A normal family has a one-parameter family of wave fronts; they are equidistant from each other.

From the symplectic point of view, the normal families are Lagrangian submanifolds in the symplectic space of rays. Since an optical reflection is a symplectic transformation, normal families are transformed to normal families. This is the classic Malus theorem; see [8] for a modern account.

Conversely, given two generic local normal families consisting of the outgoing and the incoming rays, there is a one-parameter family of mirrors that reflect one family to the other. This is Levi-Civita's theorem [9]. The mirrors are the loci of points for which the sum of distances to the respective wave fronts is constant.

For example, in dimension two, if the two normal families consist of lines through points A and B, then the respective mirrors are the ellipses with the foci A and B. Identify the circles centered at A and B with the projective line via stereographic projections. Then the respective mappings of the normal families are Möbius transformations, see Appendix and [10].

In general, it is an interesting open problem to describe the one-parameter family of mappings of normal families given by a one-mirror reflection. Another problem, motivated by applications, is as follows.

Consider a diffeomorphism of two normal families of rays in $\mathbb{R}^{n+1}$. We wish to realize this diffeomorphism as the composition of a number of mirror reflections. In particular, what is the least number of mirrors needed?

In this paper we consider a particular case of this problem: the two normal families consist of two parallel and identically oriented bundles of rays in $\mathbb{R}^{n+1}$. We think of a system of mirrors that takes a parallel beam to a parallel beam as a periscope.

In Section 2, we show that the diffeomorphism of parallel beams realized by a two-mirror reflection is a gradient diffeomorphism and, conversely, if the diffeomorphism is gradient, it can be realized by two mirrors. We describe the mirrors explicitly: they form a 2-parameter family determined by the diffeomorphism.

In Section 3, we consider the case $n = 2$. Then we have a diffeomorphism between two compact domains in $\mathbb{R}^2$. We show that if the diffeomorphism is orientation reversing then it is a composition of two gradient diffeomorphisms, and hence it can be realized by a four-mirror reflection. We present an example of an orientation preserving diffeomorphism that is not the composition of two gradient diffeomorphisms. We show that orientation preserving diffeomorphisms can be realized by six-mirror reflections. As a consequence, a diffeomorphism between two normal families of rays in $\mathbb{R}^3$ can be realized by an at most seven-mirror reflection.

In the last Section 4, we present a collection of open problems on realization of diffeomorphisms by mirror reflections.

The literature on freeform mirrors is substantial, and we mention here but a few relevant papers.

The papers [11,12] concern two-reflector systems that transform an incoming planar wave front in Euclidean space into outgoing planar wave front with a prescribed output intensity. This problem is formulated and solved as a mass transfer problem. The observation that a two-mirror reflection defines a gradient diffeomorphism, which is a part of our Theorem 1, is made in these papers: equations (4.43) in [11] and (2.3) in [12].

The paper [13] concerns mirror realizations of a mapping of a 2-parameter family of rays in $\mathbb{R}^3$ to another such family; the families are not necessarily normal. Using the Cartan–Kähler theorem in the theory of exterior differential systems, a numerical method is described for constructing four mirrors that realize the mapping (the presented examples involve only normal families of rays).

The recent paper [14] concerns a version of the question which symplectic transformations of the space of oriented lines are realized by consecutive mirror reflections. Consider a double-mirror system consisting of two infinitesimally close mirrors (thin film). A ray of light goes through the first mirror, reflects in the second one, then reflects in the first one, and escapes by going through the second mirror. This defines an infinitesimal symplectic transformation of the space of rays, that is, a Hamiltonian vector field. These Hamiltonian vector fields are described in [14] in terms of the geometry of the thin film.

Another application of mirror transformations of normal families is related to the phenomenon of invisibility, where the light rays go round a certain domain in Euclidean space (called an *invisible body*), while the corresponding transformation of a wave front is the identity (that is, the light rays are not preserved as a result of mirror reflections). A review of results on billiard invisibility can be found in chapter 8 of [5]; see also the recent paper [15] where a 2D body invisible for arbitrarily many parallel flows is constructed.

Finally, let us mention another problem of mirror design presented in [16,17], where the existence of billiard tables with locally linearizable dynamics is studied.

2. 2-mirror transformations

We are concerned with the following question: given two parallel bundles of light rays in the same direction and given a (smooth) one-to-one correspondence between the rays of the former bundle and the rays of the latter one, is it possible to realize this correspondence by means of several mirror reflections? If the answer is yes, what is the minimum number of mirrors and/or mirror reflections needed?

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