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Reduction of pre-Hamiltonian actions

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ABSTRACT

We prove a reduction theorem for the tangent bundle of a Poisson manifold (M, π) endowed with a pre-Hamiltonian action of a Poisson–Lie group (G, π_G). In the special case of a Hamiltonian action of a Lie group, we are able to compare our reduction to the classical Marsden–Ratiu reduction of M. If the manifold M is symplectic and simply connected, the reduced tangent bundle is integrable and its integral symplectic groupoid is the Marsden–Weinstein reduction of the pair groupoid $M \times \bar{M}$.

1. Introduction

Reduction procedures for manifolds with symmetries are known in many different settings. A quite general approach, whose origin traces back to the ideas of Cartan [1], was considered in [2] and then generalized in [3,4]. In this approach, the reduction of a symplectic manifold (M, ω) is intended as a submersion $\rho : N \to M_{red}$ of an immersed submanifold $i : N \hookrightarrow M$ onto another symplectic manifold (M_{red}, ω_{red}) such that $i^*\omega = \rho^*\omega_{red}$. In particular M_{red} might be the space of leaves of the characteristic distribution of $i^*\omega$. However, the most famous result is the one provided by Marsden and Weinstein [5] in the special case where the submanifold N consists of a level set of a momentum map associated to the Hamiltonian action of a Lie group. One of the possible generalizations has been introduced by Lu [6] and concerns actions of Poisson–Lie groups on symplectic manifolds. Afterwards, the case of Poisson–Lie groups acting on Poisson manifolds has been studied in [7]. It is also worth to mention the case of Manin pairs (that include Dirac and Poisson manifolds as special cases) treated in [8] and the supergeometric setting studied in [9]. In this paper we consider the case of Poisson–Lie groups acting on Poisson manifolds. Such actions appear naturally in the study of R matrices and they encode the hidden symmetries of classical integrable systems. An action of a Poisson–Lie group (G, π_G) is said to be Poisson Hamiltonian if it is generated by an equivariant momentum map $J : M \to G^*$. We shall focus on a further generalization of Poisson Hamiltonian actions. The main idea, introduced by Ginzburg in [10], is to consider only the infinitesimal version of the equivariant momentum map studied by Lu. An action induced by such an infinitesimal momentum map is what we call pre-Hamiltonian. Any Poisson Hamiltonian action is pre-Hamiltonian. Conversely, we show that any pre-Hamiltonian action is a Poisson action. However, pre-Hamiltonian actions are strictly more general of Hamiltonian

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actions, as shown in [10] where concrete examples of pre-Hamiltonian actions which are not Poisson Hamiltonian were provided.

We obtain a reduction theorem for the tangent bundle of a Poisson manifold endowed with a pre-Hamiltonian action of a Poisson–Lie group. First, we show that a pre-Hamiltonian action of a Poisson–Lie group (G, π_G) on a Poisson manifold (M, π) defines a coisotropic submanifold C of the tangent bundle TM. Hence, we can build up a reduced space by using the theory of coisotropic reduction. In fact, given a coisotropic submanifold C of TM, the associated characteristic distribution allows us to define a leaf space $C\big/\sim$, which we denote by $(TM)_{red}$. The coisotropic submanifold C is defined by means of a map $\tilde{\varphi}$ from $\mathfrak{g}$ to the space of 1-forms on M which preserves the Lie algebra structures and is a cochain map. We obtain the following

Theorem 1.1. *Let $\Phi : G \times M \to M$ be a pre-Hamiltonian action of a Poisson–Lie group G on a Poisson manifold (M, π). Then the reduced tangent bundle $(TM)_{red}$ carries a Poisson structure. Moreover $(TM)_{red} \to M/G$ is a Lie algebroid.*

The proof uses the theory of coisotropic reduction, the Tulczyjew's isomorphisms [11,12] and the theory of tangent derivations [13].

Given a pre-Hamiltonian action, we compute explicitly the infinitesimal generator of the tangent lift of the action, as can be seen in the following

Theorem 1.2. *Let $\Phi : G \times M \to M$ be a pre-Hamiltonian action of a Poisson–Lie group with infinitesimal momentum map $\tilde{\varphi}$.*

(i) *The infinitesimal generator φ^T of the tangent lift of Φ is given by*

$$\varphi^T(\xi) = X_{i_T\tilde{\varphi}_\xi} + \pi^\sharp_{TM} \circ i_T \,\mathrm{d}\,\tilde{\varphi}_\xi .$$

(ii) *If for each $\xi \in \mathfrak{g}$, one has $\mathrm{d}\,\tilde{\varphi}_\xi = 0$, then the lifted (infinitesimal) action on (TM, π_{TM}) is Hamiltonian, with fiberwise-linear momentum map defined by $c_\xi = i_T\tilde{\varphi}_\xi$.*

Note that in the special case of a symplectic action on a symplectic manifold we always obtain an Hamiltonian action on the tangent bundle and hence a Marsden–Weinstein reduction of TM can be performed.

In order to relate our reduction to the classical Marsden–Ratiu reduction, we consider the particular case of a Hamiltonian action and prove the following

Theorem 1.3. *Let (M, π) be a Poisson manifold endowed with an Hamiltonian action of a Lie group G and $0 \in \mathfrak{g}^*$ a regular value of the momentum map J. Then there is a connection dependent isomorphism of vector bundles*

$$(TM)_{red}|_{M_{red}} \cong T(M_{red}) + \tilde{\mathfrak{g}},$$

where $\tilde{\mathfrak{g}}$ is the associated bundle to the principal bundle $J^{-1}(0) \to J^{-1}(0)/G$ by the adjoint action of G on $\mathfrak{g}$.

Furthermore, by using [14,15] we provide an interpretation of the reduced tangent bundle in terms of symplectic groupoids. In particular, we consider the case of a symplectic action of a Lie group G on a symplectic manifold M. On the one hand, we show that in this case the lifted action on the tangent bundle TM is Hamiltonian so that we obtain a reduced tangent bundle $(TM)_{red}$ which is a symplectic manifold. On the other hand, it follows from [16] that the symplectic action on (M, ω) can be lifted to an Hamiltonian action on the corresponding symplectic groupoid that can be identified with the fundamental groupoid $\Pi(M) \rightrightarrows M$ of M. This implies that the symplectic groupoid can be reduced via Marsden–Weinstein procedure to a new symplectic groupoid $(\Pi(M))_{red} \rightrightarrows M/G$. We prove that in this case our reduced tangent bundle $(TM)_{red}$ is the Lie algebroid corresponding to the reduced symplectic groupoid $(\Pi(M))_{red}$. More precisely,

Theorem 1.4. *Given a free and proper symplectic action of a Lie group G on a symplectic manifold (M, ω), we have*

$$A((\Pi(M))_{red}) \cong (TM)_{red}.$$

If M is simply connected, this is just the reduction of the pair groupoid $M \times \bar{M}$.

2. Hamiltonian actions and coisotropic reduction

In this section we recall some well-known results regarding reduction procedures for Hamiltonian actions and for the more general case of coisotropic submanifolds which will be used in the following sections.

Let G be a Lie group and (M, π) a Poisson manifold. An action $\Phi : G \times M \to M$ is said to be **canonical** if it preserves the Poisson structure π on M. Let $\varphi : \mathfrak{g} \to \Gamma(TM)$ be the infinitesimal generator of the action. In order to perform a reduction we need to introduce the notion of momentum map.

Definition 2.1. A **momentum map** for a canonical action of G on M is a map $J : M \to \mathfrak{g}^*$ such that it generates the action by

$$\varphi(\xi) = \pi^\sharp(\mathrm{d} J_\xi),$$

where $J_\xi : M \to \mathbb{R}$ is defined by $J_\xi(p) = \langle J(p), \xi\rangle$, for any $p \in M$ and $\xi \in \mathfrak{g}$.

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