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Moduli spaces of semitoric systems

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ABSTRACT

Recently Pelayo–Vũ Ngọc classified simple semitoric integrable systems in terms of five symplectic invariants. Using this classification we define a family of metrics on the space of semitoric integrable systems. The resulting metric space is incomplete and we construct the completion.

1. Introduction

Toric integrable systems are classified by the image of their momentum map, which is a Delzant polytope. In [1] Pelayo–Pires–Ratiu–Sabatini define a metric on the space of Delzant polytopes via the volume of the symmetric difference and pull this back to produce a metric on the moduli space of toric integrable systems. The construction of this metric is related to the Duistermaat–Heckman measure [2].

In [3,4], Pelayo and Vũ Ngọc provide a complete classification for a broader class of integrable systems, those known as semitoric, in terms of a collection of several invariants. A *semitoric integrable system* [3] is a 4-dimensional, connected, symplectic manifold (M, ω) with a momentum map $F = (J, H) : M \to \mathbb{R}^2$ such that:

1. the function J is a proper momentum map for a Hamiltonian circle action on M;
2. F has only non-degenerate singularities (as in Williamson [5]) without real-hyperbolic blocks.

Notice that though semitoric systems are required to be 4-dimensional there is much more freedom in the choice of momentum map compared to toric systems and M is not required to be compact (the non-compact toric case is treated by Karshon–Lerman [6]). Condition (2) implies that if $p \in M$ is a critical point of F then there exists some 2×2 matrix B such that $\widetilde{F} = B \circ (F - F(p))$ is given by one of three standard forms. By Eliasson [7,8] there exists a local symplectic chart (x, y, η, ξ) centered at p which puts $\widetilde{F}$ into one of the three possible singularity types:

1. transversally elliptic singularity: $\widetilde{F}(x, y, \eta, \xi) = (\eta + \mathcal{O}(\eta^2), \frac{x^2+\xi^2}{2}) + \mathcal{O}((x, \xi)^3)$;
2. elliptic–elliptic singularity: $\widetilde{F}(x, y, \eta, \xi) = (\frac{x^2+\xi^2}{2}, \frac{y^2+\eta^2}{2}) + \mathcal{O}((x, \xi, y, \eta)^3)$;
3. focus–focus singularity: $\widetilde{F}(x, y, \eta, \xi) = (x\xi + y\eta, x\eta - y\xi) + \mathcal{O}((x, \xi, y, \eta)^3)$.

A semitoric integrable system $(M, \omega, F = (J, H))$ is said to be *simple* if there is at most one focus–focus critical point in $J^{-1}(x)$ for all $x \in \mathbb{R}$. A similar (but weaker) assumption is generic according to Zung [9], that each fiber $F^{-1}(c)$ for $c \in \mathbb{R}^2$ contains at most one critical point $p \in M$. Any semitoric system has only finitely many focus–focus critical

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points (See Vũ Ngọc [10]) so we will denote them by $m_1, \ldots, m_{m_f} \in M$ and the associated singular values are denoted $c_j = F(m_j), j = 1, \ldots, m_f$. All semitoric systems studied in this article are assumed to be simple and we label them such that $J(m_1) < \cdots < J(m_{m_f})$. Suppose that $(M_i, \omega_i, F_i = (J_i, H_i))$ is a semitoric system for $i = 1, 2$. An *isomorphism of semitoric systems* is a symplectomorphism $\phi : M_1 \to M_2$ such that $\phi^*(J_2, H_2) = (J_1, f(J_1, H_1))$ where $f : \mathbb{R}^2 \to \mathbb{R}$ is a smooth function such that $\frac{\partial f}{\partial H_1}$ nowhere vanishes (it is either always strictly positive or always strictly negative). We denote by $\mathcal{T}$ the space of simple semitoric systems modulo isomorphism.

The goal of this paper is to define a metric on the space of invariants and thus induce a metric on $\mathcal{T}$, thereby addressing Problem 2.43 from Pelayo–Vũ Ngọc [11], in which the authors ask for a description of the topology of the moduli space of semitoric systems. Problems 2.44 and 2.45 in the same article are related to the closure of $\mathcal{T}$ in the moduli space of all integrable systems, so in this paper we also compute the completion of the space of invariants, which corresponds to the completion of $\mathcal{T}$, in order to lay the foundation to begin work on these problems. The main result of this paper, Theorem A, states that the function we propose is a metric on $\mathcal{T}$ and describes the completion of the space of invariants. Theorem A is stated in Section 3 after we have defined the metric.

1.1. Notation index

Here we list some of the notation used in this article:

$\mathcal{T}$	Moduli space of simple semitoric systems, Section 1
$\mathcal{T}_{m_f}$	Elements of $\mathcal{T}$ with m_f focus–focus singular points, Section 2.1
$\mathcal{T}_{m_f,\vec{k}}$	Elements of $\mathcal{T}$ in twisting index class $\vec{k} \in \mathbb{Z}^{m_f}$, Definition 3.11
$\mathcal{T}_{m_f,[\vec{k}]}$	Elements of $\mathcal{T}$ in generalized twisting index class $\vec{k} \in \mathbb{Z}^{m_f}$, Definition 3.11
$\mathcal{M}$	Semitoric lists of ingredients, Definition 2.10
$\mathcal{M}_{m_f}$	Semitoric lists of ingredients with complexity m_f, Definition 2.10
$\mathcal{M}_{m_f,\vec{k}}$	Elements of $\mathcal{M}_{m_f}$ in twisting index class $\vec{k} \in \mathbb{Z}^{m_f}$, Section 4.6
$\mathcal{M}_{m_f,[\vec{k}]}$	Elements of $\mathcal{M}_{m_f}$ in generalized twisting index class $\vec{k} \in \mathbb{Z}^{m_f}$, Definition 3.11
$\widetilde{\mathcal{M}}$	The completion of $\mathcal{M}$, Definition 3.14
$\mathrm{Polyg}(\mathbb{R}^2)$	Rational convex polygons in $\mathbb{R}^2$, Section 2.3
$\mathcal{LW}\mathrm{Polyg}_{m_f}(\mathbb{R}^2)$	Labeled weighted polygons of complexity m_f, Definition 2.2
$(\Delta, (\ell_{\lambda_j}, \epsilon_j, k_j)_{j=1}^{m_f})$	Typical element of $\mathcal{LW}\mathrm{Polyg}_{m_f}(\mathbb{R}^2)$, Definition 2.2
$\mathcal{D}\mathrm{Polyg}_{m_f}(\mathbb{R}^2)$	Labeled Delzant semitoric polygons of complexity m_f, Definition 2.6
$\mathcal{D}^{\nu,\{b_n\}_{n=0}^{\infty}}$	Metric on $\mathcal{T}$, Definition 3.13
$d^{\nu,\{b_n\}_{n=0}^{\infty}}$	Metric on $\mathcal{M}$, Definition 3.13
$d_{m_f,[\vec{k}]}^{\nu,\{b_n\}_{n=0}^{\infty}}$	Metric on $\mathcal{M}_{m_f,[\vec{k}]}$, Definition 3.12
$d_{m_f,[\vec{k}]}^{p,\nu,\{b_n\}_{n=0}^{\infty}}$	Comparison with alignment $p \in \mathcal{S}^{m_f}$, Definition 3.12
$G_{m_f} \times \mathcal{G}$	The group $\{-1,+1\}^{m_f} \times \{T^k \mid k \in \mathbb{Z}\}$, Section 2.4
$\{b_n\}_{n=0}^{\infty}$	Linear summable sequence, Definition 3.1
$\mathcal{S}_{\vec{k},\vec{k}'}^{m_f}$	Appropriate permutations for $\vec{k}, \vec{k}' \in \mathbb{Z}^{m_f}$, Definition 3.9

2. Background: The classification of semitoric integrable systems

Since it is necessary for the construction of the metric, in this section we describe in detail the five invariants which completely classify simple semitoric systems. Compact toric integrable systems are classified in terms of Delzant polytopes. In the semitoric case a polygon plays a role but the complete invariant must contain more information. Loosely speaking, the complete invariant of semitoric systems is a collection of convex polygons in $\mathbb{R}^2$ (which may not be compact) each with a finite number of distinguished points corresponding to the focus–focus singularities labeled by a Taylor series and an integer (see Fig. 1).

2.1. The number of singular points invariant

In [10, Theorem 1] Vũ Ngọc proves that any (simple or not) semitoric system has finitely many focus–focus singular points. Thus, to a system we may associate an integer $0 \leqslant m_f < \infty$ which is the total number of focus–focus points in