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# Quantum groups and functional relations for lower rank

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A detailed construction of the universal integrability objects related to the integrable systems associated with the quantum loop algebra $\mathrm{U}_q(\mathcal{L}(\mathfrak{sl}_2))$ is given. The full proof of the functional relations in the form independent of the representation of the quantum loop algebra on the quantum space is presented. The case of the general gradation and general twisting is treated. The specialization of the universal functional relations to the case when the quantum space is the state space of a discrete spin chain is described. This is a digression of the corresponding consideration for the case of the quantum loop algebra $\mathrm{U}_q(\mathcal{L}(\mathfrak{sl}_3))$ with an extension to the higher spin case.



## 1. Introduction

The modern approach to the study of quantum integrable systems is based on the concept of the transfer matrix, or transfer operator, and the main problem here is to find its eigenvalues. The most productive method to do this is the Bethe ansatz [1]. Unfortunately, it does not work for all the cases considered as integrable. More general method was invented by Baxter, see, for example, [2,3]. He proposed to consider, together with the transfer operator, an auxiliary operator, called the $Q$-operator. The transfer operator and $Q$-operator satisfy some difference equation called the Baxter's functional $TQ$-relation. Note that in the general case we have a complex consisting of several transfer operators and $Q$-operators. At present, the transfer operators and $Q$-operators are constructed as traces of the corresponding monodromy operators and $L$-operators. We call all the operators, mentioned above, the integrability objects.

It was noted by Bazhanov, Lukyanov and Zamolodchikov [4–6] that the integrability objects can be constructed from the universal $R$-matrix of the quantum group related to the quantum integrable system under consideration. In fact, in applications to the theory of quantum integrable systems we deal with a special type of quantum groups called quantum loop algebras. Here the corresponding functional relations follow from the properties of the used representations of the quantum loop algebra under consideration. The method proved to be efficient for construction of $R$-operators [7–13], monodromy operators and $L$-operators [4–6,12–18], and for the proof of functional relations [6,14,17–20].

The general notion of a quantum group was introduced by Drinfeld and Jimbo [21–23]. Roughly speaking, it is a special kind of quasitriangular Hopf algebra. The completed tensor product of two copies of a quantum group contains the invertible element, called the universal $R$-matrix, which relates its two comultiplications, see, for example [24]. The integrability objects are determined by a choice of representations for the factors of that tensor product. The choice made for the first factor determines an integrability object, and for the second factor a concrete integrable model. It is common to call the representation space of the first factor an auxiliary space, and the representation space of the second one the quantum

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space. However, the roles of the factors can be interchanged. It appeared productive to fix representation only for the first factor, see, for example [14,15,17,18,25]. We call the arising integrability objects and functional relations universal ones. The integrability objects and functional relations for a concrete integrable model are obtained from the universal ones by applying to them the corresponding representation acting on the quantum space.

In this paper we consider quantum integrable systems related to the quantum loop algebra $\mathrm{U}_q(\mathcal{L}(\mathfrak{sl}_2))$. For this case the main tool used to construct the needed representations is the Jimbo's homomorphism [26]. In the paper [17] the Jimbo's homomorphism was defined as a mapping from $\mathrm{U}_q(\mathcal{L}(\mathfrak{sl}_2))$ to $\mathrm{U}_q(\mathfrak{sl}_2)$. It appears that the formulas become simpler and the algebraic basis more transparent if one considers the Jimbo's homomorphism as a mapping from $\mathrm{U}_q(\mathcal{L}(\mathfrak{sl}_2))$ to $\mathrm{U}_q(\mathfrak{gl}_2)$. Therefore, we rederive the universal functional relations using the latter form of the Jimbo's homomorphism. Here we use a universal approach to $TQ$- and $TT$-relations proposed in the paper [18]. We find explicit expressions for the monodromy operators and $L$-operators for the spin chains of 'spin' 1/2 particles. Then, using the fusion procedure, we construct the expressions for the 'spin' 1 case. The results of the fusion procedure allow us to define integrability objects which are Laurent polynomials on some power of the spectral parameter. Finally, we specialize the functional relations to the case of spin chains defined by a choice of arbitrary finite dimensional representation for the quantum space and write them in terms of polynomial objects.

We think that the main advantage of the undertaken consideration is a possibility, in conjunction with the results of the paper [18], of a direct conjectural generalization to the case of $\mathrm{U}_q(\mathcal{L}(\mathfrak{sl}_n))$ for an arbitrary $n$. Additionally, in comparison with the paper [17], we pay more attention to the systems of particles of higher 'spin'. The expressions for the basic monodromy operator and $L$-operator for the 'spin' 1 case are quite new. Stress that we consider the case of the general gradation and general twisting.

Depending on the sense of the deformation parameter $q$, there are at least three definitions of a quantum group. According to the first definition, $q = \exp \hbar$, where $\hbar$ is an indeterminate, according to the second one, $q$ is indeterminate, and according to the third one, $q = \exp \hbar$, where $\hbar$ is a complex number. In the first case the quantum group is a $\mathbb{C}[[\hbar]]$-algebra, in the second case a $\mathbb{C}(q)$-algebra, and in the third case it is just a complex algebra. To construct integrability objects one uses trace operations on the quantum group under consideration. To define traces it seems convenient to use the third definition of the quantum group. Therefore, we define the quantum group as a $\mathbb{C}$-algebra, see, for example, the books [27,28].

We denote by $\mathcal{L}(\mathfrak{g})$ the loop Lie algebra of a finite dimensional simple Lie algebra $\mathfrak{g}$, by $\widetilde{\mathcal{L}}(\mathfrak{g})$ its standard central extension, and by $\widehat{\mathcal{L}}(\mathfrak{g})$ the Lie algebra $\widetilde{\mathcal{L}}(\mathfrak{g})$ endowed with a derivation, see, for example, the book by Kac [29]. We follow the notations introduced by Kac. Note that nowadays the symbol $\widehat{\mathcal{L}}$ is often used instead of $\widetilde{\mathcal{L}}$ and vice versa.

The symbol $\mathbb{N}$ means the set of natural numbers and the symbol $\mathbb{Z}_+$ the set of non-negative integers.

Depending on the context, the symbol '1' means the integer one, the unit of an algebra, or the unit matrix. The symbol $\otimes$ denotes the tensor product of vector spaces and algebras and the Kronecker product of matrices with commuting or noncommuting entries.

Below we use the notation

$$\kappa_q = q - q^{-1},$$

so that the definition of the $q$-deformed number can be written as

$$[\nu]_q = \frac{q^\nu - q^{-\nu}}{q - q^{-1}} = \kappa_q^{-1}(q^\nu - q^{-\nu}), \qquad \nu \in \mathbb{C}.$$

To construct integrability objects one uses spectral parameters. They are introduced by defining a $\mathbb{Z}$-gradation of the quantum loop algebra under consideration. In the case of the loop algebra $\mathrm{U}_q(\mathcal{L}(\mathfrak{sl}_2))$ considered in this paper, a $\mathbb{Z}$-gradation is determined by two integers $s_0, s_1$. We often use the notation $s = s_0 + s_1$.

## 2. Integrability objects

### 2.1. Quantum group $\mathrm{U}q(\mathfrak{gl}_2)$

To construct integrability objects one uses appropriate representations of the quantum loop algebra under consideration. For the case of the quantum loop algebra $\mathrm{U}_q(\mathcal{L}(\mathfrak{sl}_2))$ the most important way to obtain such representations is to use the Jimbo's homomorphism from $\mathrm{U}_q(\mathcal{L}(\mathfrak{sl}_2))$ to the quantum group $\mathrm{U}_q(\mathfrak{gl}_2)$. Therefore, we first remind the definition of $\mathrm{U}_q(\mathfrak{gl}_2)$ and discuss its representations, and then proceed to $\mathrm{U}_q(\mathcal{L}(\mathfrak{sl}_2))$.

#### 2.1.1. Definition

Denote by $\mathfrak{g}$ the standard Cartan subalgebra of the Lie algebra $\mathfrak{gl}_2$ and by $G_i = E_{ii}$, $i = 1, 2$, the elements forming the standard basis of $\mathfrak{g}$.[2] The root system of $\mathfrak{gl}_2$ relative to $\mathfrak{g}$ consists of two roots $\alpha$ and $-\alpha$ where $\alpha$ is defined as

$$\alpha(G_1) = 1, \qquad \alpha(G_2) = -1. \tag{2.1}$$

[2] We use the usual notation $E_{ij}$ for the matrix units.