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HARMONIC VECTOR FIELDS ON PSEUDO-RIEMANNIAN MANIFOLDS

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Abstract. The theory of harmonic vector fields on Riemannian manifolds is generalised to pseudo-Riemannian manifolds. The congruence structure of conformal gradient fields on pseudo-Riemannian hyperquadrics and Killing fields on pseudo-Riemannian quadrics is elucidated, and harmonic vector fields of these two types are classified upto congruence. A para-Kähler twisted anti-isometry is used to correlate harmonic vector fields on the quadrics of neutral signature.

1. Introduction

Attempts to apply the variational theory of harmonic maps [6] to vector fields on Riemannian manifolds foundered at an early stage when it was observed that, for a compact Riemannian manifold (M, g), and with respect to the most natural metric h on the total space TM of the tangent bundle (viz. the Sasaki metric [15]), a vector field that is a harmonic map $(M, g) \to (TM, h)$ is necessarily parallel [9, 13]. Moreover this remains the case if the vector field is only required to be a harmonic section of the tangent bundle [16]. A more interesting theory [8] emerges in the special case where the vector field has constant length and is required to be a harmonic section of the corresponding isometrically embedded sphere sub-bundle of TM. However this theory is necessarily limited, in the compact case, to manifolds of zero Euler characteristic. Thus, the prospects for using "harmonicity" as a criterion for optimality of vector fields, or more generally sections of Riemannian vector bundles, appeared limited.

In [1] it was proposed to alleviate this problem by considering a wider range of metrics on TM. More precisely, for a fixed Riemannian metric g on M, there is an associated 2-parameter family $\mathscr{CG}$ of *generalised Cheeger-Gromoll metrics* on TM:

$$\mathscr{CG} = \{h_{p,q} : p, q \in \mathbb{R}\},$$

in which $h_{0,0} = h$ (the Sasaki metric), $h_{1,1}$ is the Cheeger-Gromoll metric [4], and $h_{2,0}$ is the stereographic metric; the general definition of $h_{p,q}$ is given in (2.2) below. The family $\mathscr{CG}$ is "natural" in the sense of [11], and more significantly renders the bundle projection $TM \to M$ a Riemannian submersion with totally geodesic fibres.

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