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Eigenvalue pinching on spin^c manifolds

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ABSTRACT

We derive various pinching results for small Dirac eigenvalues using the classification of spin^c and spin manifolds admitting nontrivial Killing spinors. For this, we introduce a notion of convergence for spin^c manifolds which involves a general study on convergence of Riemannian manifolds with a principal $\mathbb{S}^1$-bundle. We also analyze the relation between the regularity of the Riemannian metric and the regularity of the curvature of the associated principal $\mathbb{S}^1$-bundle on spin^c manifolds with Killing spinors.

1. Introduction

Eigenvalue pinching on closed manifolds is an important and widely studied topic in Riemannian geometry. It gives insight into the relation between the spectrum of an operator and the topology of the manifold. One of these studied operators is the Dirac operator on spin and spin^c manifolds. For example, Ammann and Sprouse have shown in [1, Theorem 1.8] that a spin manifold M with $r(n)$ Dirac eigenvalues close to the Friedrich bound and an appropriate lower bound on the scalar curvature implies that M is diffeomorphic to a manifold of constant curvature. Here $r(n) = 1$ if $n = 2, 3$ and $r(n) = \exp(\log(2)([\frac{n}{2}] - 1)) + 1$ if $n > 3$. The limit case of the Friedrich inequality only contains spin manifolds with real Killing spinors whose geometry was described by Bär [2] after a series of partial result of Friedrich, Grunewald, Kath and Hijazi (cf. [3–8,24–28]). Hence, Ammann and Sprouse conjectured that [1, Theorem 1.8] should also be valid with a lower value $\tilde{r}(n) < r(n)$.

This problem was considered by Vargas [9]. He introduced the concept of almost Killing spinor sequences which describes a sequence of spinors together with a sequence of metrics on a spin manifold converging to a nontrivial Killing spinor. Studying the convergence of this sequence and combining it with Gromov's compactness theorem for manifolds he derived an improved version of [1, Theorem 1.8] for simply-connected spin manifolds, [9, Theorem 5.4.1].

In this paper we define almost Killing spinors sequences on the larger class of spin^c manifolds. These will be used to derive pinching results on spin^c and spin manifolds with small spinorial Laplace eigenvalues or Dirac eigenvalues close to the Friedrich bound.

After recalling the basic definitions and properties of spin^c manifolds and Killing spinors, we shortly explain how to identify spinors of different metric spin^c structures following [10] in Section 3.

In Section 4 we will define almost Killing spinor sequences on spin^c manifolds. One of the main points we need to deal with is to derive an applicable notion of convergence of spin^c manifolds. As the spin^c structure depends on an associated principal $\mathbb{S}^1$-bundle we first study the convergence of principal $\mathbb{S}^1$-bundles with connection over closed Riemannian manifolds. This leads to one of the main results of this paper.

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Theorem 4.4. *Let $(P_j, A_j)_{j\in\mathbb{N}}$ be a sequence of principal $\mathbb{S}^1$-bundles with connection over a fixed compact Riemannian manifold (M, g). For each j let Ω_j be the 2-form representing the curvature of A_j. If there is a non-negative K such that $\|\Omega_j\|_{C^{k,\alpha}} \leq K$ for all j, then for any $\beta < \alpha$ there are a principal $\mathbb{S}^1$-bundle P with a $C^{k+1,\beta}$-connection A and a subsequence, again denoted by $(P_j, A_j)_{j\in\mathbb{N}}$ together with principal bundle isomorphisms $\Phi_j : P \to P_j$ such that $\Phi_j^* A_j$ converges to A in the $C^{k+1,\beta}$-norm.*

Afterwards we define almost Killing spinor sequences on spin^c manifolds and study their convergence behavior.

In Section 5 we analyze the regularity of spin^c manifolds with Killing spinors as spin^c manifolds with a Killing spinor are, in contrast to the spin case, in general not Einstein. We show that the existence of a nontrivial Killing spinor leads to an equation for the Ricci curvature of the manifold. Using harmonic coordinates we conclude with the results of [11]:

Theorem 5.4. *Let (M, g) be a Riemannian spin^c manifold with a $C^{1,\alpha}$-metric g, $C^{l,\alpha}$-curvature form Ω on the associated principal $\mathbb{S}^1$-bundle P, $l \geq 0$, and a nontrivial Killing spinor φ. Then g is $C^{l+2,\alpha}$ in harmonic coordinates.*

Outgoing from Theorem 4.4 we define the space $\mathcal{M}^{\mathbb{S}^1}(n, \Lambda, i_0, d, K)$, see Definition 4.7, and prove in Section 6.

Proposition 6.1. *Let Λ, i_0, d, K and k be given positive real numbers, μ a given real number and n a given natural number. Let (M, g) be a spin^c manifold in $\mathcal{M}^{\mathbb{S}^1}(n, \Lambda, i_0, d, K)$. For every $\delta > 0$ there exists a positive $\varepsilon = \varepsilon(n, \Lambda, i_0, d, K, k, \mu, \delta) > 0$ such that $\lambda_k(\nabla^{\mu\,*}\nabla^{\mu}) < \varepsilon$ implies that (M, g) has $C^{1,\alpha}$-distance smaller than δ to a spin^c manifold with k linearly independent Killing spinors with Killing number μ. Furthermore, g is at least $C^{2,\alpha}$ in harmonic coordinates.*

This proposition is the basis for all pinching results in this section. For $\mu = 0$ we combine this proposition with the geometric description of spin^c and spin manifolds with parallel spinors obtained in [12,13] and [14].

Using the Schrödinger–Lichnerowicz formula we prove a similar result to Proposition 6.1 for Dirac eigenvalues which leads again to eigenvalue pinching results for Dirac eigenvalues close to the Friedrich bound. For example, we show that even resp. odd dimensional simply-connected spin^c manifolds with one resp. two Dirac eigenvalues close to the Friedrich bound are already spin. Combining this with the geometric description of spin manifolds with real Killing spinors in [2] we show that simply-connected spin^c manifolds with a specified number of Dirac eigenvalues close to the Friedrich bound are already diffeomorphic to the sphere.

As an application of our results, we show in Section 7 that using [15, Theorem 3.1], the absolute value of the Killing number of a real Killing spinor is bounded from below by a positive constant in the class of n-dimensional Riemannian manifolds with bounded Ricci-curvature and diameter and with injectivity radius bounded from below by a positive constant.

2. Spin^c manifolds and Killing spinors

For the reader's convenience we first collect some well-known facts about spin^c manifolds. For more details see [16,17] and [18].

Definition 2.1 (*spin^c Structure*)**.** Let $\xi : \widetilde{\mathrm{GL}}(n) \to \mathrm{GL}(n)$ denote the nontrivial two-fold covering of GL(n) and set

$$\begin{aligned}\widetilde{\mathrm{GL}}^c(n) = \widetilde{\mathrm{GL}}(n)\times_{\mathbb{Z}_2}\mathbb{S}^1 &\to \mathrm{GL}(n) \times \mathbb{S}^1\\ [A, u] &\mapsto (\xi(A), u^2).\end{aligned}$$

A manifold M with frame bundle $P_{\mathrm{GL}}M$ admits a *topological spin^c structure* if there is a principal $\mathbb{S}^1$-bundle P such that there exists a principal $\widetilde{\mathrm{GL}}^c$-bundle $P_{\widetilde{\mathrm{GL}}^c}M$ that is a two-fold covering of $P_{\mathrm{GL}}M \times P$ compatible with the associated two-fold group covering.

On a Riemannian manifold (M, g) a *metric spin^c structure* is the preimage of $P_{\mathrm{SO}}M \times P$ of the topological spin^c structure, where $P_{\mathrm{SO}}M$ consists of positive oriented orthonormal frames of TM. This preimage defines a principal Spin^c-bundle $P_{\mathrm{Spin}^c}M$ with

$$\mathrm{Spin}^c(n) := \mathrm{Spin}(n)\times_{\mathbb{Z}_2}\mathbb{S}^1 \subset \widetilde{\mathrm{GL}}^c(n).$$

Since any metric spin^c structure extends uniquely to a topological spin^c structure, they have the same equivalence classes.

We now introduce *spinors* on a spin^c manifold as sections of the spin^c bundle $\Sigma^c M = P_{\mathrm{Spin}^c}M \times_\delta \Sigma_n$, where $\delta : \mathrm{Spin}^c(n) \to \mathrm{GL}(\Sigma_n)$ denotes the canonical complex spin^c representation on the complex vector space Σ_n. The spin^c bundle $\Sigma^c M$ is endowed with a natural Hermitian inner product.

A connection ∇^A on $\Sigma^c M$ is determined by a lift of the Levi-Civita connection of (M, g) together with a connection 1-form A on P to $\Sigma^c M$. The Hermitian inner product and Clifford multiplication is parallel with respect to ∇^A.

The *spinorial Laplacian* is defined as $\nabla^{A\,*}\nabla^A$ where $\nabla^{A\,*}$ is the L^2-adjoint of ∇^A. On the other hand, the *Dirac operator* is defined by its action on spinors φ, given by $D^A\varphi = \sum_{i=1}^n e_i \cdot \nabla^A_{e_i}\varphi$ in any orthonormal frame $(e_1, \ldots, e_n)$. These two operators are closely related by the Schrödinger–Lichnerowicz formula.

$$(D^A)^2\varphi = \nabla^{A\,*}\nabla^A\varphi + \frac{1}{4}\mathrm{Scal}\,\varphi + \frac{i}{2}\Omega \cdot \varphi \tag{1}$$