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Connections on decorated path space bundles

Saikat Chatterjee [a], Amitabha Lahiri [b], Ambar N. Sengupta [c,*]

[a] School of Mathematics, Indian Institute of Science Education and Research, CET Campus, Thiruvananthapuram, Kerala- 695016, India

[b] S. N. Bose National Centre for Basic Sciences, Block JD, Sector III, Salt Lake, Kolkata 700098, West Bengal, India

[c] Department of Mathematics, University of Connecticut, Storrs, CT 06269, USA

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ABSTRACT

For a principal bundle $P \to M$ equipped with a connection $\bar{A}$, we study an infinite dimensional bundle $\mathcal{P}^{\mathrm{dec}}_{\bar{A}}P$ over the space of paths on M, with the points of $\mathcal{P}^{\mathrm{dec}}_{\bar{A}}P$ being horizontal paths on P decorated with elements of a second structure group. We construct parallel transport processes on such bundles and study holonomy bundles in this setting.

1. Introduction

The focus of our study is parallel transport on bundles whose elements are paths decorated with elements of a second structure group. Geometry of this type can be studied in the language of category theory but in this work we focus exclusively on differential geometric aspects. However, we shall make remarks indicating the significance of certain notions in the category theoretic development.

We begin with a connection form $\bar{A}$ on a principal G-bundle $\pi : P \to M$, where G is a Lie group, and consider first the structure

$$\pi_{\bar{A}} : \mathcal{P}_{\bar{A}}P \to \mathcal{P}M : \overline{\gamma} \mapsto \pi \circ \overline{\gamma}, \tag{1.1}$$

where $\mathcal{P}M$ is the space of smooth paths on M and $\mathcal{P}_{\bar{A}}P$ the space of $\bar{A}$-horizontal smooth paths on P. Fig. 1 illustrates this structure.

The group G acts on the space $\mathcal{P}_{\bar{A}}P$ by right translations $\overline{\gamma} \mapsto \overline{\gamma}g$, and the structure (1.1) has the essential features of a principal G-bundle. Next we introduce a Lie group H and a semidirect product $H \rtimes_\alpha G$, which serves as a 'higher' structure group. Using these we construct a *decorated bundle*

$$\pi^{d}_{\bar{A}} : \mathcal{P}^{\mathrm{dec}}_{\bar{A}}P = \mathcal{P}_{\bar{A}}P \times H \to \mathcal{P}M : (\overline{\gamma}, h) \mapsto \pi \circ \overline{\gamma}, \tag{1.2}$$

where we view each pair $(\overline{\gamma}, h)$ as an $\bar{A}$-horizontal path $\overline{\gamma}$ on P *decorated* with an element h drawn from the second structure group H. It is this structure, illustrated in Fig. 2, that is the ultimate focus of our work in this paper. The decorated bundle

* Corresponding author.
E-mail addresses: saikat.chat01@gmail.com (S. Chatterjee), amitabhalahiri@gmail.com (A. Lahiri), ambarnsg@gmail.com (A.N. Sengupta).

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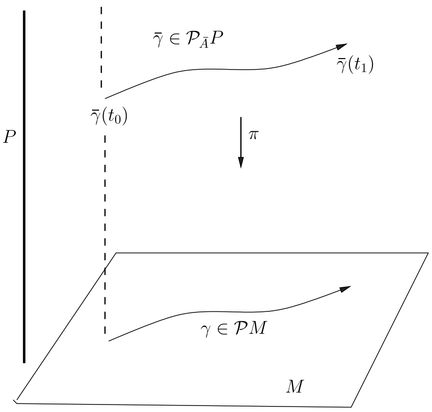

Fig. 1. Horizontal paths.

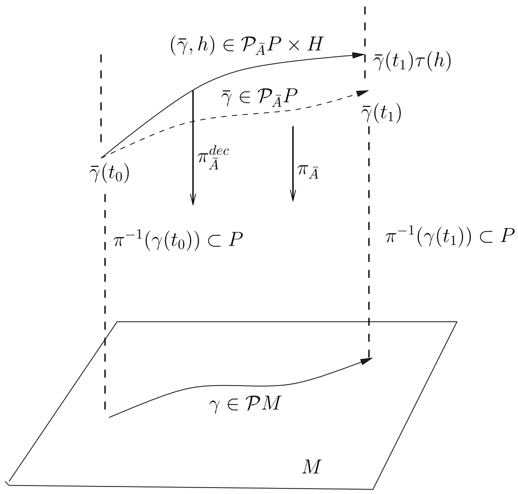

Fig. 2. Decorated paths.

arises as an example of a categorical principal bundle, as developed in [1]. Briefly put, the points of P are the objects of a category and the pairs $(\overline{\gamma}, h)$ are morphisms; the source of the morphism $(\overline{\gamma}, h)$ is the initial point $\overline{\gamma}_0$ of $\overline{\gamma}$ and the target is the point $\overline{\gamma}_1\tau(h)$, as shown in Fig. 2.

We prove results and explain how the structure (1.2) can be viewed as a principal $H \rtimes_\alpha G$-bundle. Parallel transport on this bundle takes a path on the base space $\mathcal{P}M$ of the form $[s_0, s_1] \to \mathcal{P}M : s \mapsto \Gamma_s$, and associates to it a path on the decorated bundle $\mathcal{P}_{\bar{A}}^{\mathrm{dec}}P$ of the form

$$[s_0, s_1] \to \mathcal{P}_{\bar{A}}^{\mathrm{dec}}P : s \mapsto (\hat{\Gamma}_s, h_s),$$

with a specified initial value $(\hat{\Gamma}_{s_0}, h_{s_0})$. This parallel transport process is obtained by using certain 1- and 2-forms on P with values in the Lie algebras $L(H)$ and $L(G)$. Given a suitable 1-form on P with values in the Lie algebra $L(H)$, we can associate, by a type of parallel transport process, a special element $h^*(\overline{\gamma}) \in H$ for each path $\overline{\gamma} \in \mathcal{P}_{\bar{A}}P$; this selects out an element $\left(\overline{\gamma}, h^*(\overline{\gamma})^{-1}\right) \in \mathcal{P}_{\bar{A}}^{\mathrm{dec}}P$ for each $\overline{\gamma} \in \mathcal{P}_{\bar{A}}P$. We then determine, in Section 7, conditions on the 1- and 2-forms that ensure that parallel transport of a point of $\mathcal{P}_{\bar{A}}^{\mathrm{dec}}P$ of the form $\left(\overline{\gamma}, h^*(\overline{\gamma})^{-1}\right)$ produces an element of the same type. This investigation is a study of the holonomy bundle for the decorated bundle (the holonomy bundle for a connection on a traditional finite dimensional principal bundle is a central object in the foundational work of Ambrose and Singer [2]).

The background motivation for our work arises from trying to construct a gauge theory for strings joining point particles. There is an active literature in this area, much of it focused on category theoretic aspects. In our recent works [3,1] we have developed a category theoretic framework centered on differential geometric notions such as parallel transport over spaces of decorated paths. In the present paper we establish a differential geometric development of the theory of connections over spaces of paths. For the category theoretic perspective we mention here the works of Abbaspour and Wagemann [4], Attal [5,6], Baez et al. [7,8], Barrett [9], Bartels [10], Parzygnat [11], Picken et al. [12–14], Soncini and Zucchini [15], Schreiber and Waldorf [16,17], and Wang [18,19].