Journal of Geometry and Physics 112 (2017) 197–209

Contents lists available at ScienceDirect

Journal of Geometry and Physics

journal homepage: www.elsevier.com/locate/jgp

Conformal and projective symmetries in Newtonian cosmology

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ARTICLE INFO

Article history:
Received 8 August 2016
Received in revised form 7 November 2016
Accepted 10 November 2016
Available online 21 November 2016

Keywords:
Conformal Newton–Carton group
Conformal Bargmann group
Newtonian cosmology
Newton–Hooke solution

ABSTRACT

Definitions of non-relativistic conformal transformations are considered both in the Newton–Cartan and in the Kaluza–Klein-type Eisenhart/Bargmann geometrical frameworks. The symmetry groups that come into play are exemplified by the cosmological, and also the Newton–Hooke solutions of Newton's gravitational field equations. It is shown, in particular, that the maximal symmetry group of the standard cosmological model is isomorphic to the 13-dimensional conformal-Newton–Cartan group whose conformal-Bargmann extension is explicitly worked out. Attention is drawn to the appearance of independent space and time dilations, in contrast with the Schrödinger group or the Conformal Galilei Algebra.

1. Introduction

One of the cardinal properties of the standard Friedmann–Lemaître–Robertson–Walker (FLRW) model of relativistic cosmology is its maximal conformal symmetry, arising from the vanishing of the Weyl conformal curvature. The purpose of the present article is to investigate the maximal symmetry group of an older, often unjustly dismissed, description of the non-relativistic universe at a large scale, namely Newtonian cosmology, briefly reviewed in Section 2. We found it natural to relate this search to the symmetries of general time dependent mechanical systems in the framework of Newton–Cartan (NC) geometry [1].

The surprising feature of a non-relativistic spacetime structure is the very specific independence of the NC connection with respect to the degenerate Galilei metric of spacetime [1–3]. This requires that, unlike conformal Lorentzian symmetries, *conformal-NC symmetries* be specified on the one hand by the preservation of the space and time directions of the Galilei "metric", and, on the other hand, by the preservation of the projective structure associated with the symmetric NC connection ruling free fall. This procedure, spelled out in Section 3, has proven critical in devising a purely geometrical definition [4,5] of the otherwise well-known *Schrödinger group* (re)discovered in the quantum framework in the early seventies [6,7] (see also [8] for an account about the (pre)history of the Schrödinger symmetry).

The main upshot of this article is the proof that the standard Newtonian cosmological model admits a 13-dimensional group of (local) *conformal-NC symmetries* isomorphic to the formerly named "chronoprojective group" [4,9,10] of the flat

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http://dx.doi.org/10.1016/j.geomphys.2016.11.012
0393-0440/

canonical NC structure (see Section 3.2) contains the 12-dimensional (centreless) Schrödinger group as a subgroup and is *different* from the usual centrally extended Schrödinger group.

We prove, *en passant*, that the Cosmological Principle here is strictly equivalent to the projective flatness of the Newtonian cosmological model under investigation.

Likewise, the conformal-NC group of the model of empty space with a cosmological constant is shown to be a 13-dimensional which contains the Newton–Hooke group [11], as shown in Section 3.2. The case of the vacuum static solution of the Newton field equations is treated along the same lines, highlighting the 5-dimensional "Virial group" as the conformal-NC group.

In the second part of the article, namely in Section 4, we have recourse to the so-called *Eisenhart-lift* [12–14] of a NC structure. The latter allows us to deal properly with $(\mathbb{R},+)$-extensions of non-relativistic conformal symmetry groups mentioned above. This uses the full machinery of Bargmann extended spacetime structures [15] above NC structures. In a nutshell, a Bargmann manifold is defined by a Lorentz metric and a null parallel vector field ξ, and is thus akin to a (generalized) pp-wave [16].

Those structures enable us to easily redefine the notion of non-relativistic conformal symmetries as *bona fide* conformal sub-symmetries of the Lorentz metric. This time, we introduce *conformal-Bargmann flatness* via the vanishing of the conformal Weyl curvature, $C = (C_{\lambda\mu\nu\rho})$, subject to the condition

$$C_{\lambda\mu\nu\rho}\,\xi^{\rho} = 0 \tag{1.1}$$

put forward some time ago [5,10,17]. The previously studied models are thus readily Eisenhart-lifted to yield their *conformal-Bargmann symmetries* as *non-central* extensions of their conformal-NC ones.

2. Newtonian cosmology

2.1. Newton–Cartan structures

Let us first recall that the quadruple $(N, \gamma, \theta, \nabla)$ is a *Newton–Cartan structure* if N is a smooth 4-dimensional manifold (spacetime), γ a nowhere vanishing twice-contravariant tensor, θ a closed 1-form generating the kernel of γ, and ∇ a symmetric affine connection that parallel transports γ and θ; see, e.g., [2,3,8,18]. The "clock", θ, descends to the time axis, $T = N/\ker(\theta)$. As to the tensor γ, it gives rise to a Riemannian metric on the fibres of the projection $M \to T$, that is on all copies of space at fixed time.[1] Following [3], we call such a triple (N, γ, θ) a *Galilei structure*.

For our present purpose, N will be an open subset of $\mathbb{R}^4$, and $\gamma = \delta^{ij}\,\partial_i \otimes \partial_j$ and $\theta = dt$ where $t = x^4$ is the absolute time, in a Galilean coordinate system $(x^\alpha) = (x^1, x^2, x^3, x^4)$; spatial indices $i, j, k, \ldots$ will run from 1 to 3. The NC connection, ∇, of interest to us is given by its only non-vanishing coefficients

$$\Gamma^i_{44} = -g^i \tag{2.1}$$

which represent the components of the gravitational acceleration, $\mathbf{g}$, in the chosen coordinate system.

The NC gravitational field equations [3] are given geometrically in terms of the curvature tensor, R, of ∇ by[2]

$$R_{\alpha\beta} = (4\pi G\varrho + \Lambda)\,\theta_\alpha\theta_\beta \qquad \& \qquad \gamma^{\kappa\gamma} R^{\delta}_{\alpha\kappa\beta} = \gamma^{\kappa\delta} R^{\gamma}_{\beta\kappa\alpha} \tag{2.2}$$

for all $\alpha, \beta, \gamma, \delta = 1, \ldots, 4$. The Ricci tensor, Ric, has components $R_{\alpha\beta} = R^{\kappa}_{\kappa\alpha\beta}$. In (2.2), we denote by ϱ the mass density of the sources and by Λ the cosmological constant. These equations imply that $R_{ij} = 0$, i.e., space at time t is Ricci-flat, hence (locally) flat.

In view of (2.1) and (2.2), we have

$$\boldsymbol{\nabla}\cdot\mathbf{g} = -(4\pi G\varrho + \Lambda) \qquad \& \qquad \boldsymbol{\nabla}\times\mathbf{g} = 0 \tag{2.3}$$

where $\boldsymbol{\nabla}\cdot$ and $\boldsymbol{\nabla}\times$ are the standard divergence and curl on $\mathbb{R}^3$.

2.2. Newtonian cosmology: a review

One can trace back the beginning of Newtonian cosmology to work of Kelvin [19]. We will also refer to [20–23] for the original modern expositions of non-relativistic cosmology. See also the recent contributions [24,25] using homothetic solutions of the N-body problem.

The Cosmological Principle

Following [26,4], we formulate the cosmological hypothesis of spatial homogeneity and isotropy by demanding that the curvature tensor be rotation-invariant for some freely chosen origin, $\mathbf{x} = 0$, namely that its nonzero components be given by

$$R^i_{j44} = -\partial_j g^i = a\,\delta^i_j \tag{2.4}$$

for all $i, j = 1, 2, 3$, and for some function a of time.

[1] In this section devoted to cosmology, we restrict consideration to 4-dimensional NC structures; almost all geometric definitions and results can be easily generalized to any spacetime dimension $d+1$.

[2] Our convention is $R^{\delta}_{\alpha\beta\gamma} = 2\partial_{[\alpha}\Gamma^{\delta}_{\beta]\gamma} + 2\Gamma^{\delta}_{\kappa[\alpha}\Gamma^{\kappa}_{\beta]\gamma}$.