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Fractal spectral triples on Kellendonk's C^*-algebra of a substitution tiling

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ABSTRACT

We introduce a new class of noncommutative spectral triples on Kellendonk's C^*-algebra associated with a nonperiodic substitution tiling. These spectral triples are constructed from fractal trees on tilings, which define a geodesic distance between any two tiles in the tiling. Since fractals typically have infinite Euclidean length, the geodesic distance is defined using Perron–Frobenius theory, and is self-similar with scaling factor given by the Perron–Frobenius eigenvalue. We show that each spectral triple is θ-summable, and respects the hierarchy of the substitution system. To elucidate our results, we construct a fractal tree on the Penrose tiling, and explicitly show how it gives rise to a collection of spectral triples.

1. Introduction

A tiling of the plane is a covering of $\mathbb{R}^2$ by a collection of compact subsets, called *tiles*, for which two distinct tiles can only meet along their boundaries. The building blocks of a tiling are the *prototiles*: a finite set of tiles with the property that every other tile is a translation of some prototile. A tiling is said to be *nonperiodic* if it lacks any translational periodicity. One method of producing tilings is via a substitution rule; a rule that expands each tile, and breaks it into smaller pieces, each of which is an isometric copy of an original tile. A nonperiodic substitution rule gives rise to a dynamical system, called the *continuous hull*, that consists of all tilings whose local patterns appear in some finite substitution of a prototile. The continuous hull becomes a dynamical system where the homeomorphism is induced by translation. In order to associate a particularly tractable C^*-algebra to a nonperiodic tiling, Kellendonk [1,2] places punctures in each tile, which he then uses to define a discrete subset of the continuous hull, which we refer to as the *discrete hull*.

In this paper, we define spectral triples on Kellendonk's C^*-algebra A_{punc} associated to a tiling. The fundamental new ingredients for these spectral triples, are the recently developed fractal dual substitution tilings [3]. Suppose T is a nonperiodic substitution tiling with finite prototile set $\mathcal{P}$. For each prototile $p \in \mathcal{P}$, a fractal dual tiling defines a fractal tree, in fact infinitely many, on a self-similar tiling T_p; a tiling constructed from the substitution rule on p. Each of our fractal trees defines a unique fractal path between the punctures of any two tiles in T_p. Moreover, each fractal tree on T_p respects the hierarchy of the substitution rule. Given a fractal tree on T_p, we apply Perron–Frobenius theory to the substitution matrix associated to the edges of the fractal dual tiling, to define a length function on each fractal edge in the fractal tree. This extends

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to a self-similar length function on the entire fractal tree, with scaling factor given by the Perron–Frobenius eigenvalue κ. If λ is the scaling factor for the original tiling, the scaling factor κ of the fractal tree, is related to the Hausdorff dimension h of the fractal dual tiling by the formula $h = \frac{\ln \kappa}{\ln \lambda}$. The fractal tree is then used to define a length function between any two tiles of T_p using Perron–Frobenius theory. Let $d_{\mathfrak{F}_p}(t, t')$ denote the fractal length between the punctures of two tiles t and t' in T_p.

To each substitution tiling with a fractal dual tiling, we construct spectral triples on Kellendonk's C^*-algebra A_{punc}, which we now outline. For each $p \in \mathcal{P}$, let $H_p := \ell^2(T_p \setminus \{p\})$, with canonical basis $\{\delta_t : t \in T_p \setminus \{p\}\}$, and define an unbounded multiplication operator $D_p\delta_t := \ln(d_{\mathfrak{F}_p}(t, p))\delta_t$. We show that (A_{punc}, H_p, D_p) is a θ-summable (positive) spectral triple. Let $H := \oplus_{p\in\mathcal{P}} H_p$. For each function $\sigma : \mathcal{P} \to \{-1, 1\}$, we define an unbounded multiplication operator $D_\sigma := \oplus_{p\in\mathcal{P}} \sigma(p)D_p$. Then, (A_{punc}, H, D_σ) is also a θ-summable spectral triple. This defines a collection of spectral triples on Kellendonk's algebra A_{punc} that each respect the hierarchy of the substitution rule.

Using operator algebras as the basic framework, Alain Connes developed noncommutative geometry [4], and has shown its significance to many fields of mathematics. In particular, one of the overarching themes of noncommutative geometry is to describe a consistent mathematical model for quantum physics. Dynamical systems are particularly well suited to the tools of noncommutative geometry, and provide dynamical invariants in a noncommutative framework. Of particular importance to Connes' program are spectral triples, which typically define a noncommutative Riemannian metric on a C^*-algebra. A spectral triple (A, H, D) consists of a C^*-algebra A faithfully represented on a separable Hilbert space H, and a self-adjoint unbounded operator D on H with compact resolvent, whose commutators with a dense $*$-subalgebra of A are bounded.

The noncommutative topology of tilings has a long history. Alain Connes initiated the study of substitution tilings in a noncommutative framework by giving a detailed description of a C^*-algebra associated with the Penrose tiling in his seminal book [4]. In 1982, Dan Shechtman discovered quasicrystals [5], a type of material that is neither crystalline nor amorphous. The mathematical theory explaining Shechtman's discovery had already been developed in the context of purely mathematical research; nonperiodic tilings provide an excellent model for quasicrystals. In an attempt to understand the physics of quasicrystals, Bellissard defined a crossed product C^*-algebra by a family of Schrödinger operators [6,7]. Years later, Kellendonk defined a discrete version of the continuous hull and constructed a groupoid C^*-algebra associated with a tiling [1,8]. Soon afterwards, Anderson and Putnam [9] showed that the continuous hull Ω of a tiling is a Smale space, and used this observation to describe the K-theory of the crossed product $C(\Omega) \rtimes \mathbb{R}^d$. More recently, Kellendonk's construction was generalised to tilings with infinite rotational symmetry in [10], and the rotationally equivariant K-theory of these algebras was completely worked out in [11].

Only recently has there been a breakthrough in the noncommutative geometry of tilings. The primary interest in spectral triples on tilings is that the continuous hull of a nonperiodic tiling is not only a topological object, it also has rich geometric structure. The groundbreaking spectral triple for tilings appeared in John Pearson's 2008 thesis [12], and the subsequent joint paper with Bellissard [13]. These spectral triples were defined on the commutative C^*-algebra associated with the hull of a tiling. A few years later, the second author constructed spectral triples on the unstable C^*-algebra of a Smale space [14,15], which is strongly Morita equivalent to Kellendonk's algebra. However, in the special case of substitution tiling algebras, this spectral triple essentially measured the Euclidean distance between two tilings in the groupoid used to define the C^*-algebra, and ignored the substitution rule. Since Bellissard and Pearson's seminal result there have been a number of papers on spectral triples of tilings, see for example [16–20]. The survey article [21] explains these constructions and their relationship to one another.

2. Nonperiodic tilings and their properties

The tilings in this paper are built from *prototiles*, a finite collection $\mathcal{P}$ of labelled compact subsets of $\mathbb{R}^2$ that each contain the origin, and are equal to the closure of their interior. We denote the label of a prototile $p \in \mathcal{P}$ by $\ell(p)$, the support of p by $\text{supp}(p) \subset \mathbb{R}^2$, and the boundary of p by $\partial\,\text{supp}(p)$. In general, given a subset $X \subset \mathbb{R}^2$, we write ∂X for the boundary of X. The labels allow us to have two distinct prototiles with the same support, and we often denote the labels by colours. A *tile* is defined to be any translation of a prototile. So, for any $p \in \mathcal{P}$ and $x \in \mathbb{R}^2$, the labelled subset $t := p + x$ is a tile with label $\ell(t) := \ell(p)$, and support $\text{supp}(t) := \text{supp}(p) + x$.

Definition 2.1. Let $\mathcal{P}$ be a set of prototiles. A *tiling of the plane* is a countable collection T of tiles, each of which is a translate of some prototile $p \in \mathcal{P}$, such that

(1) $\cup_{t\in T}\ \text{supp}(t) = \mathbb{R}^2$; and
(2) $\text{int}(\text{supp}(t)) \cap \text{int}(\text{supp}(t')) = \varnothing$ whenever $t \neq t'$.

A tiling T is said to be *edge-to-edge* if whenever two tiles intersect, they meet full edge to full edge, or at a common vertex (see [3, Section 3.2] for the definition of an edge in the case that the tiling does not have polygonal prototiles). If tiles in an edge-to-edge tiling T only intersect along at most one edge, then T is said to be *singly edge-to-edge*. A *patch* $P \subset T$ is a finite collection of tiles in T such that the interior of the support of P is connected.

For $x \in \mathbb{R}^2$ and $r > 0$, let $B(x, r)$ denote the ball of radius r centred at x. Given a tiling T, we require the following collections of tiles. For $x \in \mathbb{R}^2$ and $r > 0$, let

$$T \sqcap B(x, r) := \{t \in T : t \subset B(x, r)\}.$$