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Quantum heat traces

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ABSTRACT

We study new invariants of elliptic partial differential operators acting on sections of a vector bundle over a closed Riemannian manifold that we call the relativistic heat trace and the quantum heat traces. We obtain some reduction formulas expressing these new invariants in terms of some integral transforms of the usual classical heat trace and compute the asymptotics of these invariants. The coefficients of these asymptotic expansion are determined by the usual heat trace coefficients (which are locally computable) as well as by some new global invariants.

1. Introduction

The heat kernel is one of the most important tools of global analysis, spectral geometry, differential geometry and mathematical physics [1–6], in particular, quantum field theory, even financial mathematics [7]. In quantum field theory the main objects of interest are described by the Green functions of self-adjoint elliptic partial differential operators on manifolds and their spectral invariants such as the functional determinants. In spectral geometry one is interested in the relation of the spectrum of natural elliptic partial differential operators to the geometry of the manifold. There are also non-trivial links between the spectral invariants and the non-linear completely integrable evolution systems, such as Korteweg–de Vries hierarchy (see, e.g. [4,8–10]). In many interesting cases such systems are, in fact, infinite-dimensional Hamiltonian systems, and the spectral invariants of a linear elliptic partial differential operator are nothing but the integrals of motion of this system. In financial mathematics the behavior of the derivative securities (options) is determined by some deterministic parabolic partial differential equations of diffusion type with an elliptic partial differential operator of second order. The conditional probability density is then nothing but the fundamental solution of this equation, in other words, the heat kernel [7].

Instead of studying the spectrum of a differential operator directly one usually studies its spectral functions, that is, spectral traces of some functions of the operator, such as the zeta function, and the heat trace. Usually one does not know the spectrum exactly; that is why, it becomes very important to study various asymptotic regimes. It is well known, for example, that one can get information about the asymptotic properties of the spectrum by studying the short time asymptotic

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expansion of the heat trace. The coefficients of this expansion, called the heat trace coefficients (or global heat kernel coefficients), play very important role in spectral geometry and mathematical physics [3,4].

The existence of non-isometric isospectral manifolds demonstrates that the spectrum alone does not determine the geometry (see, e.g. [11]). That is why, it makes sense to study more general invariants of partial differential operators, maybe even such invariants that are not spectral invariants, that is, invariants that depend not only on the eigenvalues but also on the eigenfunctions, and, therefore, contain much more information about the geometry of the manifold.

The case of a Laplace operator on a compact manifold without boundary is well understood and there is a vast literature on this subject, see [3] and the references therein. In this case there is a well defined local asymptotic expansion of the heat kernel, which enables one to compute its diagonal and then the heat trace by directly integrating the heat kernel diagonal; this gives all heat trace coefficients. The heat trace asymptotics of Laplace type operators have been extensively studied in the literature, and many important results have been discovered. The early developments are summarized in the books [2,3] with extensive bibliography, see also [9,12].

We initiate the study of new invariants of second-order elliptic partial differential operators acting on sections of vector bundles over compact Riemannian manifolds without boundary. The long term goal of this project is to develop a comprehensive methodology for such invariants in the same way as the theory of the standard heat trace invariants. We draw a deep analogy between the spectral invariants of elliptic operators and the classical and quantum statistical physics [13]. We consider an elliptic self-adjoint positive partial differential operator H and its square root, $\omega = H^{1/2}$, which is an elliptic self-adjoint positive *pseudo*-differential operator of first order.

In Section 2 we motivate the study of the invariants the *relativistic heat trace*

$$\Theta_r(\beta) = \mathrm{Tr}\ \exp(-\beta\omega) \tag{1.1}$$

and the *quantum heat traces*

$$\Theta_b(\beta, \mu) = \mathrm{Tr}\,\{\exp[\beta(\omega-\mu)] - 1\}^{-1}, \tag{1.2}$$

$$\Theta_f(\beta, \mu) = \mathrm{Tr}\,\{\exp[\beta(\omega-\mu)] + 1\}^{-1}, \tag{1.3}$$

where Tr denotes the standard L^2 trace, β is a positive parameter and μ is a (generally, non-positive) parameter. We would like to warn the reader who might have questions about this definition that a detailed discussion will appear in Section 2. An invariant similar to our relativistic heat trace, namely, the heat trace of a square root of a Laplace type operator, was studied in [6]. We also introduce the corresponding zeta functions: the *relativistic zeta function*

$$Z_r(s, \mu) = \frac{1}{\Gamma(s)} \int_0^\infty d\beta\ \beta^{s-1} e^{\beta\mu}\, \Theta_r(\beta) \tag{1.4}$$

and the *quantum zeta functions*

$$Z_{b,f}(s, \mu) = \frac{1}{\Gamma(s)} \int_0^\infty d\beta\ \beta^{s-1} \Theta_{b,f}(\beta, \mu). \tag{1.5}$$

We show that these new invariants can be reduced to some integrals of the well known classical heat trace

$$\Theta(t) = \mathrm{Tr}\ \exp(-tH) \tag{1.6}$$

and compute the asymptotics of these invariants as $\beta \to 0$.

In Section 3 we review the standard theory of the heat kernel of the Laplace type operator H in a form suitable for our analysis. To be precise, we consider a closed Riemannian manifold M of dimension n, a vector bundle $\mathcal{V}$ over M and an elliptic self-adjoint second-order *partial* differential operator H with a positive definite scalar leading symbol of Laplace type acting on sections of the bundle $\mathcal{V}$. We introduce a function A_q of a complex variable q defined by

$$A_q = (4\pi)^{n/2} \frac{1}{\Gamma(-q)} \int_0^\infty dt\ t^{-q-1+n/2}\, \Theta(t). \tag{1.7}$$

Then we show that for a positive operator H the function A_q is entire and its values at non-negative integer points $q = k$ are equal to the standard heat trace coefficients, which are locally computable, while the values of the function A_q at the points $q = k + 1/2$, with k an integer, as well as the values of its derivative at the positive integer points are new global invariants that are not locally computable.

We would like to warn the reader about our nomenclature. We call the real numbers of the form $k + 1/2$, with k an integer, *half-integers*. Note that a half-integer is *not* a half *of* an integer since a half of an even integer is an integer and not a half-integer. That is, in our nomenclature, half-integers are a half of odd integers, which are sometimes also called *half-odd-integers*. We hope that our use of the term half-integer should not cause a confusion.

In Section 4 we compute the relativistic zeta function $Z_r(s, \mu)$ and the asymptotics of the relativistic heat trace $\Theta_r(\beta)$ as $\beta \to 0$. We obtained in even dimension $n = 2m$,

$$\Theta_r(\beta) \sim \sum_{k=0}^{\infty} \beta^{2k-2m} b_k^{(1)} A_k + \sum_{k=0}^{\infty} \beta^{2k+1} b_k^{(2)} A_{k+m+1/2}, \tag{1.8}$$