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Invertible linear ordinary differential operators

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ABSTRACT

We consider invertible linear ordinary differential operators whose inversions are also differential operators. To each such operator we assign a numerical table. These tables are described in the elementary geometrical language. The table does not uniquely determine the operator. To define this operator uniquely some additional information should be added, as it is described in detail in this paper. The possibility of generalization of these results to partial differential operators is also discussed.

1. Introduction

We describe invertible linear differential operators whose inversions are also linear differential operators. There are many various applications of such operators [1, § 2.3]. Transformations and classification of systems of differential equations (or, more generally, differential objects) are among them. Classical transformations of systems of differential equations are invertible changes of dependent and independent variables of the systems. They are referred to as Lie transformations [2, Chap. 4]. Their theory has been developed quite completely. However, there are invertible transformations such that the variables of one system depend both on the variables of the other system and on derivatives of dependent variables with respect to independent ones. Such transformations are called $\mathcal{C}$-transformations [2, Chap. 6] or Lie–Backlund transformations [3]. In particular, invertible nonlinear differential operators may be understood as $\mathcal{C}$-transformations [2, item 6.3.8]. In the case of a single dependent variable, any $\mathcal{C}$-transformation is a Lie transformation. However, this fails in the case of several independent variables. Moreover, in this case, there are much more $\mathcal{C}$-transformations than Lie transformations.

To use $\mathcal{C}$-transformations it is necessary to have their convenient description. A $\mathcal{C}$-transformation of linear systems is an invertible linear differential operator. In the case of nonlinear systems, the linearization of a $\mathcal{C}$-transformation can be interpreted as an invertible linear differential operator [4]. Therefore, the study of invertible linear differential operators should be understood as the first step to the description of $\mathcal{C}$-transformations of both linear and nonlinear systems.

The problem of describing $\mathcal{C}$-transformations has become especially important during the last 20 years in connection with the development of the theory of flat control systems. Flat systems are defined as systems equivalent to linear controllable systems with respect to the group of $\mathcal{C}$-transformations (see [5]). Control methods developed for linear systems are generalized to flat systems. Numerous papers (see the bibliography in [6]) show that such systems describe various natural phenomena and processes. Therefore, the characterization of flat systems is of a considerable interest.

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Invertible linear differential operators with one independent variable are used to characterize flat systems in [7–9]. This approach is related to the deformation theory of pseudogroup structures [10,11]. This relationship is investigated in [4]. As a result, the deformation theory is generalized to the case of infinite-dimensional manifolds. Vector fields whose local flows consist of $\mathcal{C}$-transformations are studied in [12]. Invertible linear differential operators with two independent variables are described in [13]. In particular it is proved that any two-sided invertible operator can be written as a composition of triangular invertible operators in some stable sense.

In this paper we describe invertible linear differential operators with one independent variable. Our approach is similar to the approach in [14] and is based on assigning a numerical table to each invertible operator. These tables are further described using an elementary geometrical language. Thus, to each invertible operator one assigns an elementary geometric model, which is referred to as a d-scheme.

An invertible linear differential operator is not uniquely determined by its d-scheme. Below we show how to construct an invertible differential operator for a given d-scheme and what structures should be still given for constructing. The proofs of the main results rely on a description of d-schemes in the language of spectral sequences.

This paper is organized as follows. In Section 1, we define invertible linear differential operators with one independent variable and present their examples. We state elementary geometric models of such operators in Section 2. The main results of the present paper are given in Section 3, and their proof is contained in Section 4. Corollaries and generalizations of these results are discussed at the end of the paper.

2. Invertible linear differential operators

Let M be a one-dimensional manifold, let $\mathcal{A} = C^\infty(M)$ be the $\mathbb{R}$-algebra of smooth functions on M, and let $\mathcal{P}$ and $\mathcal{Q}$ be modules of smooth sections of some vector bundles ξ and ζ over M.

Recall that if t is a coordinate on M and $p_1, \dots, p_m$ are coordinates in fibers of the vector bundle ξ, then any section of ξ may be represented as a column of functions $p(t) = (p_1(t), \dots, p_m(t))^T$. Similar representation exists for sections of ζ. Denote by $q_1, \dots, q_{m_0}$ coordinates in fibers of ζ.

Let k be some nonnegative integer. A map Δ of $\mathcal{P}$ in $\mathcal{Q}$ is called a *linear differential operator* of order $\le k$ (or simply a *differential operator*) if in coordinates,

$$\Delta\big(p(t)\big) = (q_1(t), \dots, q_{m_0}(t))^T, \quad q_j(t) = \sum_{i=1}^{m}\sum_{l=0}^{k} a_{jil}(t)\frac{d^l p_i(t)}{dt^l}, \; a_{jil} \in \mathcal{A}, \tag{1}$$

where m and m_0 are the dimensions of ξ and ζ respectively. This definition is a coordinate version of a definition from [15, Chap. 9, § 2]. The algebraic definition of differential operators can be found in [2, item 0.2.2]. For our purposes, it is more convenient to use the coordinate definition.

Denote by ord Δ the *order* of a linear differential operator Δ, i.e., $k = \operatorname{ord}\Delta$ iff Δ is an operator of order $\le k$ but is not an operator of order $\le k-1$.

The set of all linear differential operators of order $\le k$ acting from $\mathcal{P}$ into $\mathcal{Q}$ is an $\mathcal{A}$-module under the multiplication $(a^+\Delta)(p) = \Delta(ap)$, $p \in \mathcal{P}$. Denote by $\mathrm{Diff}_k^+(\mathcal{P}, \mathcal{Q})$ this module. The map $\Delta \mapsto \Delta(1)$ establishes an isomorphism of the module $\mathrm{Diff}_0^+(\mathcal{A}, \mathcal{Q})$ to the module $\mathcal{Q}$. Besides, from the definition it follows that $\mathrm{Diff}_k^+(\mathcal{P}, \mathcal{Q}) \subset \mathrm{Diff}_{k+1}^+(\mathcal{P}, \mathcal{Q})$ for any $k \ge 0$. The set $\mathrm{Diff}_*^+(\mathcal{P}, \mathcal{Q}) = \cup_{k=0}^\infty \mathrm{Diff}_k^+(\mathcal{P}, \mathcal{Q})$ is an $\mathcal{A}$-module of infinite dimension.

A linear differential operator of $\mathcal{P} = \mathcal{A}$ to $\mathcal{Q} = \mathcal{A}$ is called *scalar*. Since composition of scalar linear differential operators is a scalar linear differential operator again, the set $\mathrm{Diff}_*^+(\mathcal{A}, \mathcal{A})$ is a noncommutative ring.

A linear differential operator $\Delta\colon \mathcal{P} \to \mathcal{Q}$ is called (*two-sided*) *invertible* if there exists a linear differential operator $\Delta^{-1}\colon \mathcal{Q} \to \mathcal{P}$ such that the composition $\Delta^{-1} \circ \Delta$ is the identity mapping of the module $\mathcal{P}$ and the composition $\Delta \circ \Delta^{-1}$ is the identity mapping of the module $\mathcal{Q}$. In this case, the operator Δ^{-1} is called the *inversion* of Δ.

It is convenient to represent elements of $\mathcal{P}$, $\mathcal{Q}$ as columns of functions, and a linear differential operator $\Delta\colon \mathcal{P} \to \mathcal{Q}$ as a matrix of scalar operators. It can be proved that any invertible linear differential operator is represented by a square matrix.

In the following two examples, a, b, c, d are arbitrary scalar linear differential operators.

Example 1. In the case $m = 3$, the linear differential operators Δ and Δ^{-1} determined by the matrices

$$\Delta = \begin{pmatrix} 1 & -a & 0 \\ -c & ca+1 & -b \\ dc & -dca-d & db+1 \end{pmatrix}, \qquad \Delta^{-1} = \begin{pmatrix} ac+1 & abd+a & ab \\ c & bd+1 & b \\ 0 & d & 1 \end{pmatrix}$$

are inverse. This can be proved by matrix multiplication.

Example 2. In the case $m = 2$, the operators Δ and Δ^{-1} determined by the matrices

$$\Delta = \begin{pmatrix} 1+bc & -b \\ -a-c-abc & 1+ab \end{pmatrix}, \qquad \Delta^{-1} = \begin{pmatrix} 1+ba & b \\ a+c+cba & 1+cb \end{pmatrix}$$

are also inverse.