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Nonlinear first order PDEs reducible to autonomous form polynomially homogeneous in the derivatives

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ABSTRACT

It is proved a theorem providing necessary and sufficient conditions enabling one to map a nonlinear system of first order partial differential equations, polynomial in the derivatives, to an equivalent autonomous first order system polynomially homogeneous in the derivatives. The result is intimately related to the symmetry properties of the source system, and the proof, involving the use of the canonical variables associated to the admitted Lie point symmetries, is constructive. First order Monge–Ampère systems, either with constant coefficients or with coefficients depending on the field variables, where the theorem can be successfully applied, are considered.

1. Introduction

Lie group analysis [1–8] provides a unified and elegant algorithmic framework to a deep understanding and fruitful handling of differential equations. It is known that Lie point symmetries admitted by ordinary differential equations allow for their order lowering and possibly reducing them to quadrature, whereas in the case of partial differential equations they can be used for the determination of special (invariant) solutions of initial and boundary value problems. Also, the Lie symmetries are important ingredients in the derivation of conserved quantities, or in the construction of relations between different differential equations that turn out to be equivalent [8–16]. Lie point symmetries of differential equations, in fact, can be used to construct a mapping from a given (source) system of differential equations to another (target) suitable system; if we consider one-to-one (invertible) point mappings, then a one-to-one correspondence between Lie point symmetries admitted by the source and target system of differential equations arises. In other words, the Lie algebra of infinitesimal operators of the target system of differential equations has to be isomorphic to the Lie algebra of infinitesimal operators of the source system of differential equations. This property has been used to give necessary and sufficient conditions for reducing a system of partial differential equations to autonomous form [12,13], a system of first order nonlinear partial differential equations to linear form [9–11,13], nonautonomous and/or nonhomogeneous quasilinear systems of partial differential equations to autonomous and homogeneous form [14,16]. In particular, in [16], it has been proved a theorem providing the necessary and sufficient conditions in order to map a general first order quasilinear system of partial differential equations,

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say

$$\sum_{i=1}^{n} A^i(\mathbf{x},\mathbf{u})\frac{\partial \mathbf{u}}{\partial x_i} = \mathbf{B}(\mathbf{x},\mathbf{u}), \tag{1}$$

where $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{u} \in \mathbb{R}^m$, A^i are $m \times m$ matrices with entries depending at most on $\mathbf{x}$ and $\mathbf{u}$, and the source term $\mathbf{B} \in \mathbb{R}^m$ depends at most on $\mathbf{x}$ and $\mathbf{u}$ too, into a first order quasilinear homogeneous and autonomous system. This reduction, when it is possible, is performed by an invertible point transformation like

$$\mathbf{z} = \mathbf{Z}(\mathbf{x}), \qquad \mathbf{w} = \mathbf{W}(\mathbf{x},\mathbf{u}), \tag{2}$$

which preserves the quasilinear structure of the system, and whose construction is algorithmically suggested by the Lie symmetries admitted by (1).

In this paper, we consider a general nonlinear system of first order partial differential equations involving the derivatives of the unknown variables in polynomial (of degree greater than 1) form, and establish a theorem giving necessary and sufficient conditions in order to map it to an autonomous system which is polynomially homogeneous in the derivatives.

In some relevant situations, *e.g.*, Monge–Ampère systems, the target system results to be quasilinear, but there are cases where the system we obtain is polynomially homogeneous in the derivatives but not quasilinear. This means that the conditions of the theorem are only necessary for the reduction of a *nonlinear* first order system to autonomous and homogeneous *quasilinear* form [17].

The main difference of the theorem presented in this paper with the similar one proved in [16] (concerned with the transformation of a general first order quasilinear system of partial differential equations into a first order quasilinear homogeneous and autonomous system) consists in the possibility of admitting now an invertible point transformation like

$$\mathbf{z} = \mathbf{Z}(\mathbf{x},\mathbf{u}), \qquad \mathbf{w} = \mathbf{W}(\mathbf{x},\mathbf{u}), \tag{3}$$

i.e., a mapping where the new independent variables $\mathbf{z}$ are allowed to depend also on the old dependent ones.

The plan of the paper is the following. In Section 2, the theorem giving necessary and sufficient conditions for the existence of an invertible mapping linking a nonlinear system of first order partial differential equations which is polynomial in the derivatives to an autonomous system polynomially homogeneous in the derivatives is proved. In Section 3, the theorem is applied to various general first order Monge–Ampère systems. Finally, Section 4 contains some concluding remarks.

2. Main result

Let us consider a general system of first order partial differential equations

$$\Delta\left(\mathbf{x},\mathbf{u},\mathbf{u}^{(1)}\right) = 0, \tag{4}$$

where $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{u} \in \mathbb{R}^m$ and $\mathbf{u}^{(1)} \in \mathbb{R}^{mn}$ are the independent variables, the dependent variables, and the first order partial derivatives, respectively. In particular, in the following we consider systems (4) composed by equations which are polynomial in the derivatives, with coefficients depending at most on $\mathbf{x}$ and $\mathbf{u}$, *i.e.* systems made by equations of the form

$$\sum_{|\boldsymbol{\alpha}|,|\mathbf{j}|=1}^{N_s} A^s_{\boldsymbol{\alpha}\mathbf{j}}(\mathbf{x},\mathbf{u})\prod_{k=1}^{|\boldsymbol{\alpha}|}\frac{\partial u_{\alpha_k}}{\partial x_{j_k}} + B^s(\mathbf{x},\mathbf{u}) = 0, \qquad s = 1,\dots,m, \tag{5}$$

where $\boldsymbol{\alpha}$ is the multi-index $(\alpha_1,\dots,\alpha_r)$, $\mathbf{j}$ the multi-index $(j_1,\dots,j_r)$, $\alpha_k = 1,\dots,m$, $j_k = 1,\dots,n$, N_s are integers, and $A^s_{\boldsymbol{\alpha}\mathbf{j}}(\mathbf{x},\mathbf{u})$, $B^s(\mathbf{x},\mathbf{u})$ smooth functions of their arguments.

The aim is to determine necessary and sufficient conditions for the construction of an invertible point transformation

$$\mathbf{z} = \mathbf{Z}\left(\mathbf{x},\mathbf{u}\right), \qquad \mathbf{w} = \mathbf{W}\left(\mathbf{x},\mathbf{u}\right), \tag{6}$$

mapping (5) into an equivalent autonomous system which is homogeneous polynomial in the derivatives $\mathbf{w}^{(1)}$, *i.e.*, made by equations of the form

$$\sum_{|\boldsymbol{\alpha}|,|\mathbf{j}|=\overline{N}_s} \widetilde{A}^s_{\boldsymbol{\alpha}\mathbf{j}}(\mathbf{w})\prod_{k=1}^{\overline{N}_s}\frac{\partial w_{\alpha_k}}{\partial z_{j_k}} = 0, \qquad s = 1,\dots,m, \tag{7}$$

for some integers $\overline{N}_s$; of course, it may occur that the target system turns out to be linear in the derivatives, *i.e.*, $\overline{N}_s = 1$ ($s = 1,\dots,m$), whereupon we have an autonomous and homogeneous quasilinear system.

The following lemma, guarantees that an invertible point transformation like (6) preserves the polynomial structure in the derivatives.

Lemma 1. *Given a first order system of partial differential equations like (5) which is polynomial in the derivatives, then an invertible point transformation like (6) produces a first order system which is still polynomial in the derivatives.*

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