ARTICLE IN PRESS

Journal of Geometry and Physics ▮ (▮▮▮▮) ▮▮▮–▮▮▮

Contents lists available at ScienceDirect

Journal of Geometry and Physics

journal homepage: www.elsevier.com/locate/jgp

Nonlocal conservation laws of the constant astigmatism equation

Adam Hlaváč, Michal Marvan *

Mathematical Institute in Opava, Silesian University in Opava, Na Rybníčku 1, 746 01 Opava, Czech Republic

ARTICLE INFO

Article history:
Received 17 March 2016
Accepted 9 June 2016
Available online xxxx

MSC:
37K05
37K25
37K35

Keywords:
Constant astigmatism equation
Conservation laws
Abelian covering
Reciprocal transformation
Harry Dym equation

ABSTRACT

For the constant astigmatism equation, we construct a system of nonlocal conservation laws (an abelian covering) closed under the reciprocal transformations. The corresponding potentials are functionally independent modulo a Wronskian type relation.

1. Introduction

The constant astigmatism equation [1]

$$z_{yy} + \left(\frac{1}{z}\right)_{xx} + 2 = 0 \tag{1}$$

represents surfaces characterised by constant difference between the principal radii of curvature, with x, y being the curvature coordinates. The same equation represents orthogonal equiareal patterns on the unit sphere, closely related to sphere's plastic deformations within itself, see [2]. Both topics are classical, see, e.g., Bianchi [3, § 375], although Eq. (1) itself did not appear at that time.

It is clear from the geometry that the constant astigmatism equation is transformable to the sine-Gordon equation $\phi_{XY} = \sin\phi$ [2]. Then, of course, the constant astigmatism equation itself is integrable in the sense of soliton theory. Known are the zero curvature representation, see [1] or Eq. (7), the bi-Hamiltonian structure and hierarchies of higher order symmetries and conservation laws [4], as well as multi-soliton solutions [5].

* Corresponding author.
E-mail addresses: Adam.Hlavac@math.slu.cz (A. Hlaváč), Michal.Marvan@math.slu.cz (M. Marvan).

http://dx.doi.org/10.1016/j.geomphys.2016.06.002
0393-0440/

Please cite this article in press as: A. Hlaváč, M. Marvan, Nonlocal conservation laws of the constant astigmatism equation, Journal of Geometry and Physics (2016), http://dx.doi.org/10.1016/j.geomphys.2016.06.002

According to our previous paper [6], the constant astigmatism equation has six local conservation laws with associated potentials $\chi, \xi, \eta, \zeta, \alpha, \beta$ satisfying

$$\begin{aligned}
\chi_x &= z_y + y, & \chi_y &= \frac{z_x}{z^2} - x, \\
\eta_x &= x z_y, & \eta_y &= x\frac{z_x}{z^2} + \frac{1}{z} - x^2, \\
\xi_x &= -y z_y + z - y^2, & \xi_y &= -y\frac{z_x}{z^2}, \\
\zeta_x &= x y z_y - x z + \frac{1}{2} x y^2, & \zeta_y &= x y\frac{z_x}{z^2} + \frac{y}{z} - \frac{1}{2} x^2 y, \\
\alpha_x &= \frac{\sqrt{(z_x + z z_y)^2 + 4z^3}}{z}, & \alpha_y &= \frac{\sqrt{(z_x + z z_y)^2 + 4z^3}}{z^2}, \\
\beta_x &= \frac{\sqrt{(z_x - z z_y)^2 + 4z^3}}{z}, & \beta_y &= -\frac{\sqrt{(z_x - z z_y)^2 + 4z^3}}{z^2}.
\end{aligned}$$

Potentials α, β correspond to the independent variables X, Y of the related sine-Gordon equation [1].

Potentials ξ, η are images of x, y under the reciprocal transformations [6] $\mathcal{Y}, \mathcal{X}$, respectively; see formulas (15). Applying $\mathcal{X}$ to ξ and $\mathcal{Y}$ to η, we obtain new nonlocal potentials and the process can be continued indefinitely. It is then natural to ask what is the minimal set of potentials closed under the action of $\mathcal{X}$ and $\mathcal{Y}$. They are also nonlocal conservation laws of the sine-Gordon equation, but available descriptions [7,8] are not of much help.

It has been known from the very beginning of the soliton theory that hierarchies of conservation laws arise through expansion in terms of the spectral parameter [9]. Literature on the subject is vast and many ways to connect integrability and hierarchies of conservation laws have been proposed (see, for example, [10,11] or [12, Prop. 1.5] or [13, Sect. 5d] or [14]).

The constant astigmatism equation belongs to the rare cases when nonlocal conservation laws can be obtained in a very straightforward way, almost effortlessly. We then look how they are acted upon by the reciprocal transformations $\mathcal{X}$ and $\mathcal{Y}$ and look for existing relations among the potentials. A provably functionally independent set of potentials is presented.

2. Preliminaries

Let $\mathcal{E}$ be a system of partial differential equations in two independent variables x, y. A conservation law is a 1-form $f\,\mathrm{d}x + g\,\mathrm{d}y$ such that $f_y - g_x = 0$ as a consequence of the system $\mathcal{E}$. A *potential*, say ϕ, corresponding to this conservation law is a variable which formally satisfies the compatible system $\phi_x = f, \phi_y = g$.

Let $\mathfrak{g}$ be a matrix Lie algebra. A $\mathfrak{g}$-valued *zero curvature representation* [15] of the system $\mathcal{E}$ is a 1-parametric family of $\mathfrak{g}$-valued forms $\alpha(\lambda) = A(\lambda)\,\mathrm{d}x + B(\lambda)\,\mathrm{d}y$ such that $A_y - B_x + [A, B] = 0$ as a consequence of the system $\mathcal{E}$.

Let Q be an arbitrary matrix (called a *gauge matrix*) belonging to the associated Lie group $\mathcal{G}$. The *gauge transformation* [15] with respect to Q sends $\alpha = A\,\mathrm{d}x + B\,\mathrm{d}y$ to ${}^Q\alpha = {}^QA\,\mathrm{d}x + {}^QB\,\mathrm{d}y$, where

$$ {}^QA = Q_x Q^{-1} + QAQ^{-1}, \qquad {}^QB = Q_y Q^{-1} + QBQ^{-1}. \tag{2}$$

We also say that ${}^QA\,\mathrm{d}x + {}^QB\,\mathrm{d}y$ is *gauge equivalent* to $A\,\mathrm{d}x + B\,\mathrm{d}y$.

The shortest way to conservation laws is from a zero curvature representation that vanishes at some value λ_0 of λ. Without loss of generality we assume that $\lambda_0 = 0$, i.e., $A(0) = B(0) = 0$. Consider the associated compatible linear system [15] (or a differential covering [16,17])

$$\Phi_x = A\Phi, \qquad \Phi_y = B\Phi, \tag{3}$$

where Φ is a column vector. Expanding Φ into the formal power series

$$\Phi = \sum_{i=0}^{\infty} \Phi_i \lambda^i$$

around zero and inserting into (3), we obtain compatible equations

$$\Phi_{n,x} = \sum_{i=1}^{n} A_i \Phi_{n-i}, \qquad \Phi_{n,y} = \sum_{i=1}^{n} B_i \Phi_{n-i}, \quad n \ge 0, \tag{4}$$

where A_i, B_i are the coefficients of the Taylor expansion of A, B around $\lambda = 0$. Here we start from $i = 1$ since $A_0 = B_0 = 0$. By formulas (4), each of the derivatives $\Phi_{n,x}, \Phi_{n,y}$ is explicitly expressed in terms of $\Phi_0, \dots, \Phi_{n-1}$. Moreover, $\Phi_{0,x} = \Phi_{0,y} = 0$, meaning that Φ_0 is a constant vector. Choosing Φ_0 suitably, we thus obtain what may be called a hierarchy of vectorial