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Conformal differential invariants

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ABSTRACT

We compute the Hilbert polynomial and the Poincaré function counting the number of fixed jet-order differential invariants of conformal metric structures modulo local diffeomorphisms, and we describe the field of rational differential invariants separating generic orbits of the diffeomorphism pseudogroup action. This resolves the local recognition problem for conformal structures.

0. Introduction

Differential invariants play a central role in the classification problems of geometric structures. Often the fundamental invariants have tensorial character, but for resolution of the equivalence problem scalar invariants are required to be derived from those.

For instance, the fundamental invariant of a Riemannian metric g on a manifold M is the Riemann curvature tensor $R_g \in \Gamma(\Lambda^2 T^*M \otimes \mathfrak{so}(TM))$. Scalar differential invariants are Weyl curvature invariants [1], separating generic orbits of the diffeomorphism pseudogroup $G = \mathrm{Diff}_{\mathrm{loc}}(M)$ acting on the space of jets of metrics $J^\infty(S^2_{\mathrm{ndg}} T^*M)$, where $S^2_{\mathrm{ndg}} T^*M$ is the complement in $S^2 T^*M$ to the cone of degenerate quadrics, and they are obtained by contractions of the tensor products of the covariant derivatives of the curvature tensor R_g. Their number depending on the jet-order was computed by Zorawski [2] and Haskins [3], see also [4].

In this paper we do the same for conformal metric structures $(M, [g])$ of arbitrary signature in dimensions $n = \dim M > 2$. Notice that for $n = 2$ the conformal group is too large and, due to Gauß theorem on existence of isothermal coordinates, there are no local invariants of conformal structures, and hence no differential invariants in 2D.

The fundamental invariants C of the conformal structure are the Cotton tensor for $n = 3$ and the Weyl tensor for $n > 3$. Similarly to Weyl scalar invariants for Riemannian metrics, one could expect scalar invariants to be derived from the fundamental tensor invariants, and this was done in [5–7], and will be discussed in the next section. These scalar invariants are however defined on the (proper jet-lift of the) ambient space $\hat{M}$ to our M, $\dim \hat{M} = n+2$, so that the constructed scalars are covariants rather than invariants.

There is however an easy approach to construct differential invariants for generic conformal structures. It is based on the folklore result that in the domain $U \subset M$, where $\|C\|_g^2 \neq 0$ for some (and hence any) representative $g \in [g]$, one can uniquely fix (actually up to $\pm$ if the signature is split) a metric g_0 in the conformal class $[g]$ by the normalization $\|C\|_{g_0}^2 = \pm 1$ (the sign is always $+$ in the Riemannian case, but can be any in the indefinite case). Then the conformal invariants are derived from the (pseudo-)Riemannian metric ones (Weyl curvature invariants or those from [8,9]).

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This however does not yield the number[1] of scalar differential invariants $H_n(k)$ depending on the jet-order k (we count so-called "pure order", see below). The classical approach to computing these numbers is the Lie method of elimination of group parameters (or algebra parameters), see [3,2,10]. This involves calculation of ranks of large matrices. Instead we rely on some simple algebraic ideas and compute the Hilbert polynomial $H_n(k)$, the first values of which are given below:

$n \setminus k$	1	2	3	4	...	k
3	0	0	1	9	...	k^2-4
4	0	3	36	91	...	$\frac{1}{6}(k+2)(k+3)(5k-7)$
5	0	24	135	350	...	$\frac{1}{24}(k+2)(k+3)(k+4)(9k-11)$

Then we derive the Poincaré function encoding these numbers. We also indicate a different set of conformal differential invariants, now rational, and describe the field they generate.

1. The algebras and fields of differential invariants

The scalar conformal invariants mentioned in the introduction are constructed via the ambient metric construction of Fefferman and Graham [5] roughly as follows. Consider the bundle $\bar{M} = M \times \mathbb{R}_+$ over M consisting of all representatives g of $[g]$ with its natural horizontal metric $\bar{g}$ (tautological structure: $\bar{g}_g = g \circ d_g\pi$, where $\pi : \bar{M} \to M$), and let $\hat{M} = \bar{M} \times (-1, 1)$. The ambient metric $\hat{g}$ is $\mathbb{R}_+$-scaling weight 2 homogeneous Ricci flat Lorentzian metric on $\hat{M}$ restricting to $\bar{g}$ on $\bar{M} \times \{0\}$. This exists on the infinite jet of $\bar{M} \times \{0\} \subset \hat{M}$ for odd n, and up to order $n/2$ for even n. Taking the Weyl metric curvature invariants of $\hat{g}$ yields scalar invariants of $[g]$, which give a complete set of polynomial invariants[2] for odd n and the same to a finite order for even n, see [6,7]. The definite advantage of these invariants is that they are defined for all conformal structures.

There are however two basic problems with these ambient Weyl conformal invariants, similar to the classical Weyl metric curvature invariants. First of all, the algebra generated by these polynomial invariants is not finitely generated. Secondly, it is not a priori clear which of these differential invariants are separating for the orbits of the diffeomorphism pseudogroup action (on infinite or any finite jet-level).

The second problem is solved by passing to rational differential invariants: since the action is algebraic, its prolongations are algebraic too [11], and in any finite jet-order there exists a rational quotient by the action due to the Rosenlicht theorem [12]. From this viewpoint the field $\mathfrak{F}$ of rational differential invariants is useful and simpler. The invariants obtained in this way will be presented below.

The first problem is a bit more complicated, as it is clear that the transcendence degree $\operatorname{trdeg}(\mathfrak{F}) = \infty$, so just passing to rational invariants does not resolve infinite generation. In the early days of differential invariants theory it was suggested and motivated by Sophus Lie and Arthur Tresse that the algebra of differential invariants is generated by a finite number of differential invariants $I_1, \dots, I_t$ and a finite number of invariant derivations $\nabla_1, \dots, \nabla_s$. This was later proved in several versions, see [11] and the references therein.

In more details, consider the algebra $\mathfrak{A}_l$ of differential invariants that are rational by the jets of order $\leq l$ and polynomial by the jets of higher order (l is determined by the structure in question, we will see that in the case of conformal structures $l = 4$ for $n = 3$ and $l = 3$ for $n > 3$). This $\mathfrak{A}_l$ is called the algebra of rational-polynomial invariants.

The main result of [11] states that $\mathfrak{A}_l$ is finitely generated by I_i, ∇_j, i.e. any differential invariant from $\mathfrak{A}_l$ is a polynomial of $\nabla_J I_i$ for ordered multi-indices $J = (j_1, \dots, j_r)$ with rational coefficients of I_k.

Now the field of rational differential invariants $\mathfrak{F}$ is generated by $\mathfrak{A}_l$ for some l, and so is also finitely generated in the Lie–Tresse sense as above. The algebra $\mathfrak{A}_l$ separates the orbits of the G-action on the space of jets of conformal structures $J^\infty(\mathcal{C}_M)$, where

$$\mathcal{C}_M = S^2_{\mathrm{ndg}} T^*M/\mathbb{R}_+,$$

and we get a finite separating set of invariants $\nabla_J I_i$, $|J| \leq k - \deg(I_i)$ for the restriction of the action on J^k. Thus we obtain a set of generators for the field $\mathfrak{F}_k$ of rational invariants of order k that filter the field $\mathfrak{F}$.

2. Scalar invariants of conformal metric structures

Let us generate conformal differential invariants of generic conformal structures $[g]$. This will in turn generate rational differential invariants on the space of jets of all conformal structures $J^\infty(\mathcal{C}_M)$.

We begin with the case $n \geq 4$. Since there are no metric invariants of order <2, there are no conformal invariants of lower order too. The lowest order conformal invariants live in 2-jets. It is well-known that the complete invariant there is

[1] We have to fix the signature (p, q) of $[g]$, $p + q = n$. The formulas for invariants vary with this (p, q), but the number of invariants depends only on n.

[2] When we write "polynomial" here and beyond we mean only with respect to jets of order >0, allowing division by the determinant of g everywhere.

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