ARTICLE IN PRESS

Journal of Geometry and Physics ▮ (▮▮▮▮) ▮▮▮–▮▮▮

Contents lists available at ScienceDirect

Journal of Geometry and Physics

journal homepage: www.elsevier.com/locate/jgp

Differential invariants of self-dual conformal structures

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A R T I C L E I N F O

Article history:
Received 5 May 2016
Accepted 18 May 2016
Available online xxxx

Keywords:
Differential invariants
Invariant derivations
Self-duality
Conformal metric structure
Hilbert polynomial
Poincaré function

A B S T R A C T

We compute the quotient of the self-duality equation for conformal metrics by the action of the diffeomorphism group. We also determine Hilbert polynomial, counting the number of independent scalar differential invariants depending on the jet-order, and the corresponding Poincaré function. We describe the field of rational differential invariants separating generic orbits of the diffeomorphism pseudogroup action, resolving the local recognition problem for self-dual conformal structures.

Introduction

Self-duality is an important phenomenon in four-dimensional differential geometry that has numerous applications in physics, twistor theory, analysis, topology and integrability theory. A pseudo-Riemannian metric g on an oriented four-dimensional manifold M determines the Hodge operator $*: \Lambda^2 TM \to \Lambda^2 TM$ that satisfies the property $*^2 = \mathbf{1}$ provided g has the Riemannian or split signature. In this paper we restrict to these two cases, ignoring the Lorentzian signature.

The Riemann curvature tensor splits into $O(g)$-irreducible pieces $R_g = \mathrm{Sc}_g + \mathrm{Ric}_0 + W$, where the last part is the Weyl tensor [1] and $O(g)$ is the orthogonal group of g. In dimension 4, due to exceptional isomorphisms $\mathfrak{so}(4) = \mathfrak{so}(3) \oplus \mathfrak{so}(3)$, $\mathfrak{so}(2,2) = \mathfrak{so}(1,2) \oplus \mathfrak{so}(1,2)$, the last component splits further $W = W_+ + W_-$, where $*W_\pm = \pm W_\pm$. Metric g is called self-dual if $*W = W$, i.e. $W_- = 0$. This property does not depend on conformal rescalings of the metric $g \to e^{2\varphi} g$, and so is the property of the conformal structure $[g]$.

Since the space of W_- has dimension 5, and the conformal structure has 9 components in 4D, the self-duality equation appears as an underdetermined system of 5 PDE on 9 functions of 4 arguments. This is however a misleading count, since the equation is natural, and the diffeomorphism group acts as the symmetry group of the equation. Since $\mathrm{Diff}(M)$ is parametrized by 4 functions of 4 arguments, we expect to obtain a system of 5 PDE on $5 = 9 - 4$ functions of 4 arguments.

This 5×5 system is determined, but it has never been written explicitly. There are two approaches to eliminate the gauge freedom.

One way to fix the gauge is to pass to the quotient equation that is obtained as a system of differential relations (syzygies) on a generating set of differential invariants. By computing the latter for the self-dual conformal structures we write the quotient equation as a nonlinear 9×9 PDE system, which is determined but complicated to investigate.

Another approach is to get a cross-section or a quasi-section to the orbits of the pseudogroup $G = \mathrm{Diff}_{\mathrm{loc}}(M)$ action on the space $\mathcal{SD} = \{[g] : W_- = 0\}$ of self-dual conformal metric structures. This was essentially done in the recent work [2, III.A]: By choosing a convenient ansatz the authors of that work encoded all self-dual structures via a 3×3 PDE system

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http://dx.doi.org/10.1016/j.geomphys.2016.05.017

$\mathcal{SDE}$ of the second order (this works for the neutral signature; in the Riemannian case use doubly biorthogonal coordinates to get self-duality as a 5×5 second-order PDE system [2, III.C] that can be investigated in a similar manner as the 3×3 system).

In this way almost all gauge freedom was eliminated, yet a part of symmetry remained shuffling the structures. This pseudogroup $\mathcal{G}$ is parametrized by 5 functions of 2 arguments (and so is considerably smaller than G). We fix this freedom by computing the differential invariants of $\mathcal{G}$-action on $\mathcal{SDE}$ and passing to the quotient equation.

The differential invariants are considered in rational–polynomial form, as in [3]. This allows to describe the algebra of invariants in Lie–Tresse approach, and also using the principle of n-invariants of [4]. We count differential invariants in both approaches and organize the obtained numbers in the Hilbert polynomial and the Poincaré function.

1. Scalar invariants of self-dual structures

The first approach to compute the quotient of the self-duality equation by the local diffeomorphisms pseudogroup G action is via differential invariants of self-dual structures $\mathcal{SD}$. The signature of the metric g or conformal metric structure $[g]$ is either $(2, 2)$ or $(4, 0)$. In this and the following two sections we assume that g is a Riemannian metric on M for convenience. Consideration of the case $(2, 2)$ is analogous.

To distinguish between metrics and conformal structures we will write $\mathcal{SD}_m$ for the former and $\mathcal{SD}_c$ for the latter. Denote the space of k-jets of such structures by $\mathcal{SD}_m^k$ and $\mathcal{SD}_c^k$ respectively. These clearly form a tower of bundles over M with projections $\pi_{k,l} : \mathcal{SD}_{\rm x}^k \to \mathcal{SD}_{\rm x}^l$, $\pi_k : \mathcal{SD}_{\rm x}^k \to M$, where x is either m or c.

1.1. Self-dual metrics: invariants

Consider the bundle $S^2_+T^*M$ of positively definite quadratic forms on TM and its space of jets $J^k(S^2_+T^*M)$. The equation $W_- = 0$ in 2-jets determines the submanifold $\mathcal{SD}_m^2 \subset J^2$, and its prolongations are $\mathcal{SD}_m^k \subset J^k$ for $k > 2$.

Computation of the stabilizer of the action shows that the submanifolds $\mathcal{SD}_m^k$ are regular, meaning that generic orbits of the G-action in $\mathcal{SD}_m^k$ have the same dimension as in $J^k(S^2_+T^*M)$. This is based on a simple observation that generic self-dual metrics have no symmetry at all. Thus the differential invariants of the action on $\mathcal{SD}_m^k$ can be obtained from the differential invariants on the jet space J^k [5,6].

These invariants can be constructed as follows. There are no invariants of order ≤ 1 due to existence of geodesic coordinates, the first invariants arise in order 2 and they are derived from the Riemann curvature tensor (as this is the only invariant of the 2-jet of g). Traces of the Ricci tensor $\mathrm{Tr}(\mathrm{Ric}^i)$, $1 \leq i \leq 4$, yield 4 invariants $I_1, \dots, I_4$ that in a Zariski open set of jets of metrics can be considered horizontally independent, meaning $\hat{d}I_1 \wedge \cdots \wedge \hat{d}I_4 \neq 0$.

To get other invariants of order 2, choose an eigenbasis $e_1, \dots, e_4$ of the Ricci operator (in a Zariski open set it is simple), denote the dual coframe by $\{\theta^i\}$ and decompose $R_g = R^i_{jkl} e_i \otimes \theta^j \otimes \theta^k \wedge \theta^l$. These invariants include the previous I_i, and the totality of independent second-order invariants for self-dual metrics is

$$\dim\{R_g | W_- = 0\} - \dim O(g) = (20 - 5) - 6 = 9.$$

The invariants R^i_{jkl} are however not algebraic, but obtained as algebraic extensions via the characteristic equation. Then R^i_{jkl} (9 independent components) and e_i generate the algebra of invariants.

Alternatively, compute the basis of Tresse derivatives $\nabla_i = \hat{\partial}_{I_i}$ and express the metric in the dual coframe $\omega^j = \hat{d}I_j$: $g = G_{ij}\omega^i\omega^j$. Then the functions I_i, G_{kl} generate the space of invariants by the principle of n-invariants [4].

Remark. There is a natural almost complex structure $\hat{J}$ on the twistor space of self-dual (M, g), i.e. on the bundle $\hat{M}$ over M whose fiber at a consists of the sphere of orthogonal complex structures on T_aM inducing the given orientation. The celebrated theorem of Penrose [7,1] states that self-duality is equivalent to integrability of $\hat{J}$. Thus local differential invariants of g can be expressed through semi-global invariants of the foliation of the three-dimensional complex space $\hat{M}$ by rational curves. Similarly in the split signature one gets foliation by α-surfaces, and the geometry of this foliation of $\hat{M}$ yields the invariants on M.

We explain how to get rid of non-algebraicity in the next subsection.

1.2. Self-dual conformal structures: invariants

Here the invariants of the second order are obtained from the Weyl tensor as the only conformally invariant part of the Riemann tensor R_g. For general conformal structures a description of the scalar invariants was given recently in [8]. In our case $W = W_+ + W_-$ the second component vanishes, and so we have only 5-dimensional space of curvature tensors $\mathcal{W}$, namely Weyl parts of R_g considered as $(3, 1)$ tensors.

Let us fix a representative of the conformal structure $g_0 \in [g]$ by the requirement $\|W_+\|^2_{g_0} = 1$, this uniquely determines g_0 provided that W_+ is non-vanishing in a neighborhood (in the case of neutral signature we have to require $\|W_+\|^2_g \neq 0$